Individual Contest

1. Find three positive integers such that their pairwise least common multiples are 9800, 2200 and 539.

   Solution:
   We have $9800 = 2^3 \times 5^2 \times 7^2$, $2200 = 2^3 \times 5^2 \times 11$ and $539 = 7^2 \times 11$. Only one of the numbers is divisible by 7, only one number is divisible by 11, and only one number is divisible by 10. Therefore, the three numbers must be $7^2 = 49$, 11 and $2^3 \times 5^2 = 200$.

2. Each of eight numbers after the first is obtained by adding the same amount to the preceding number. The sum of the middle two numbers is 16. Find the sum of all eight numbers.

   Solution:
   The sum of the first number and the eighth number must be equal to the sum of the second number and the seventh number, which must in turn be equal to the sum of the third number and the sixth number, as well as to the sum of the middle two numbers. Hence the grand total is $4 \times 16 = 64$.

3. The sum of five consecutive positive integers is the cube of an integer and the sum of the middle three numbers is the square of an integer. Find the smallest possible value of the middle number.

   Solution:
   The sum of five consecutive numbers is equal to 5 times the middle number and the sum of three consecutive numbers is equal to 3 times the middle number. Hence the 5 times the middle number is a cube and 3 times the middle number is a square. It follows that the middle number is 25 times a cube and 3 times a square. The smallest possible value for this number is when the cube is $3^3 = 27$ and the square is $5^2 = 25$, yielding the number $27 \times 25 = 675$.

4. Find two four-digit numbers such that their difference is 2014 and all eight digits in these two numbers are different from one another.

   Solution:
   Clearly, there is a borrowing from the hundreds digit. Suppose there is no borrowing from the tens digit. Then the tens digits must be 0 and 9 and we need three pairs of digits differing by 1, 2 and 4 respectively. We have the following.

$$\begin{array}{r} 6207 \\ -4193 \\ \hline 2014 \end{array} \quad \begin{array}{r} 8207 \\ -6193 \\ \hline 2014 \end{array} \quad \begin{array}{r} 5208 \\ -3194 \\ \hline 2014 \end{array} \quad \begin{array}{r} 7208 \\ -5194 \\ \hline 2014 \end{array} \quad \begin{array}{r} 6305 \\ -4291 \\ \hline 2014 \end{array}$$

$$\begin{array}{r} 8305 \\ -6291 \\ \hline 2014 \end{array} \quad \begin{array}{r} 7308 \\ -5294 \\ \hline 2014 \end{array} \quad \begin{array}{r} 8405 \\ -6391 \\ \hline 2014 \end{array} \quad \begin{array}{r} 7406 \\ -5392 \\ \hline 2014 \end{array} \quad \begin{array}{r} 3506 \\ -1492 \\ \hline 2014 \end{array}$$

$$\begin{array}{r} 4607 \\ -2593 \\ \hline 2014 \end{array} \quad \begin{array}{r} 4705 \\ -2691 \\ \hline 2014 \end{array} \quad \begin{array}{r} 4805 \\ -2791 \\ \hline 2014 \end{array} \quad \begin{array}{r} 6805 \\ -4791 \\ \hline 2014 \end{array}$$

Suppose there is also a borrowing from the tens digit. Then the tens digits are either 0 and 8 or 1 and 9. We have the following.

$$\begin{array}{rrrrr} 7203 & 6203 & 6301 & 4601 & 7310 \\ -5189 & -4189 & -4287 & -2587 & -5296 \\ \hline 2014 & 2014 & 2014 & 2014 & 2014 \end{array}$$

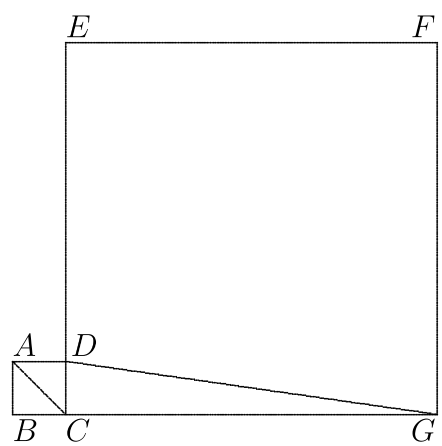
$$\begin{array}{rrrrr} 7410 & 5810 & 4810 & 7412 & 5712 \\ -5396 & -3796 & -2796 & -5398 & -3698 \\ \hline 2014 & 2014 & 2014 & 2014 & 2014 \end{array}$$

5. The middle four digits of a six-digit multiple of 33 are 2014. Find the largest possible value of such a six-digit number.

   Solution:
   The six-digit number is divisible by 3 and 11. The digit sum of 2014 is 7 and the alternate digit sum of 2014 is 1. Hence the alternate digit sum of the six-digit number must be 0, so that its digit sum must be even. The digit sum is at least 3+5+7=15 and at most 8+9+7=24. Hence it is either 18 or 24. Suppose it is 18. Then the sum of the two unknown digits is 11. Since their difference is 1, the number must be 520146. Suppose it is 24. Then the number must be 820149. This is the largest possible value.

6. $ABCD$ and $CGFE$ are squares such that G lies on the extension of BC and D lies on CE. The area of triangle GCD is 7 times the area of triangle CAD. Find the ratio of the area of triangle ABC to the quadrilateral $DEFG$.

   Solution:
   Since GCD and CAD have a common base CD, their altitudes are in the ratio $GC : AD = 7 : 1$. The areas of $DEFG$ and $ABCD$ are in the ratio $7^2 : 1^2 = 49 : 1$. Hence the areas of ABC and $DEFG$ are in the ratio $\frac{1}{2} \times 1 : \frac{2\times7-1}{2\times7} \times 49 = 1 : 91$.



7. At 9:00 am, Carla leaves the city travelling at a constant speed, and arrives at a village at 12:40 pm. Mary leaves the city an hour later, travelling at a constant speed along the same road, and arrives at the village at noon. Find the time when Mary overtakes Carla.

Solution:
Carla's trip takes 220 minutes while Mary's takes 120 minutes. Hence the ratio of their speeds is 11:6. Thus in 1 hour, Mary covers $11 - 6 = 5$ units more than Carla. With a start of 1 hour, Carla has gone ahead of Mary by 6 units. This can be made up by Mary in $\frac{6}{5}$ hours. It follows that Mary overtakes Carla at 11:12 am.

8. The sum of the reciprocals of four positive integers, not necessarily different from one another, is equal to $\frac{7}{10}$. Find the smallest possible sum of these four integers.

   Solution:
   If the smallest of these four integers is 6, then $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3} < \frac{7}{10}$. Hence the smallest one is at least 5. If it is 5 and the other three integers are 6, then $\frac{1}{5} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{7}{10}$, and the sum of the four integers is 5+6+6+6=23. Now if the sum of two numbers is fixed, the sum of their reciprocal decreases when their difference decreases. For instance, $\frac{1}{6} + \frac{1}{7} < \frac{1}{5} + \frac{1}{8}$ because $\frac{1}{7} - \frac{1}{8} = \frac{1}{56} < \frac{1}{30} = \frac{1}{5} - \frac{1}{6}$. If the sum of the four integers is 22, the minimum value of the sum of their reciprocal is $\frac{1}{5} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6} > \frac{7}{10}$. Hence 23 is the smallest possible sum of the four integers.

9. When he computes the sum of eleven consecutive positive integers, Alex very carelessly omits two consecutive numbers and gets a total of 9832. Find the correct total.

   Solution:
   Since the sum of eleven consecutive numbers is always odd, the middle number, which is equal to the average, must also be odd. The average value of the nine numbers is $9832 \div 9 = 1092\frac{4}{9}$. Suppose the middle number is 1091. Then the eleven numbers go from 1086 to 1096. Their sum is $1091 \times 11 = 12001$ and the sum of the two omitted numbers is $12001 - 9832 = 2169$. Hence they must 1084 and 1085, but they are outside the range. If the middle number is less than 1091, the situation is worse. Suppose the middle number is 1093. Then the eleven numbers go from 1088 to 1098. Their sum is $1093 \times 11 = 12023$ and the sum of the two omitted numbers is $12023 - 9832 = 2191$. Hence they must be 1095 and 1096, and these values are within the range. Suppose the middle number is 1095. Then the eleven numbers go from 1090 to 1100. Their sum is $1095 \times 11 = 12045$ and the sum of the two omitted numbers is $12045 - 9832 = 2213$. Hence they must be 1106 and 1107, which are outside the range. If the middle number is greater than 1095, the situation is worse. It follows that the correct sum must be 12023.

10. There are four problems in a mathematics contest. None of the 100 students solves all four problems. The first problem is solved by 90 students, the second is solved by 80 students, the third solved by 70 students and the fourth by 60 students. Find the number of students who solve both the third problem and the fourth problems.

    Solution:
    Note that 10 students miss the first problem and 20 students miss the second problem. So at least $100 - 10 - 20 = 70$ students solve both the first problem and the second problem. Since no student solves all four problems, at most $100 - 70$ students solve both the third problem and the fourth problem. Note that 30 students miss the third problem and 40 students miss the fourth problem. So at least 30 students solve the third problem and the fourth problem. It follows that exactly 30 students solve these two problems.

11. If the five numbers $2^{147} \times 6^{49}$, $3^{98} \times 5^{49}$, $5^{98} \times 2^{49}$, 7^{98} and $128^7 \times 23^{49}$ are arranged in ascending order, find the number in the middle.

Solution:
Taking the 49-th roots of the numbers, they reduce to 48, 45, 50, 49 and 46 respectively. Hence the number in the middle is $48^{49} = 2^{147} \times 6^{49}$.

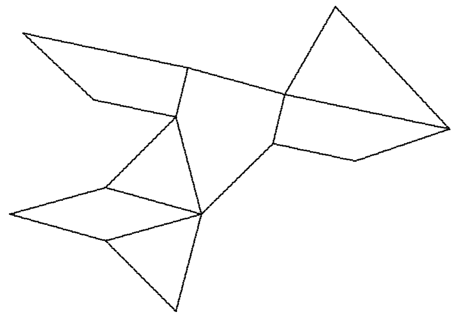
12. The positive integers 1 to 12 inclusive are arranged in a circle. The difference between the square of any number and the product of its two neighbours is divisible by 13. If the number 2 follows the number 1 in clockwise order, find the order of the remaining numbers clockwise after 2.

Solution:
The number after 2 is $2^2 = 4$ more than a multiple of 13. If this multiple is -13, the number is -9. If this multiple is 0, the number is 4. If this multiple is 13, the number is 17. Only 4 is within the range. Hence the number after 2 is 4. The number after 4, multiplied by 2, is $4^4 = 16$ more than a multiple of 13, which must be even. If this multiple is -26, the number is -5. If this multiple is 0, the number is 8. If this multiple is 26, the number is 21. Only 8 is within the range. Hence the number after 4 is 8. Similarly, the numbers after 8 are $(8^2 - 52) \div 4 = 3$, $(3^3 + 39) \div 8 = 6$, $(6^2 + 0) \div 3 = 12$, $(12^2 - 78) \div 6 = 11$, $(11^2 - 13) \div 12 = 9$, $(9^2 - 26) \div 11 = 5$, $(5^2 + 65) \div 9 = 10$ and $(10^2 - 65) \div 5 = 7$. We check that we indeed have $(7^2 - 39) \div 10 = 1$ and $(1 + 13) \div 7 = 2$, completing the circle without repetition.

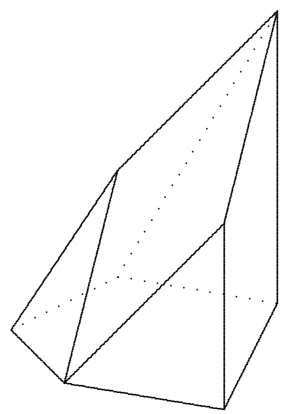
13. If Max adds six times the money Nina has to what he has, and Nina adds six times the money Max has to what she has, the product of the new amounts, in dollars, is 2014. Find the maximum amount of money, in dollars, Max can have.

Solution:
Note that $2014 = 2 \times 19 \times 53$. Since both new amounts must be greater than 2, one of them is 19 or 53, and the other is twice 19 or twice 53. To maximize the amount of money Max has, Nina's new amount should be smaller. We consider two cases.
Case 1. Nina's new amount is 19 and Max's is 106.
From Nina's new amount, Max has 1, 2 or 3 dollars, and Nina has 13, 7 or 1 dollars respectively. However, none of $6 \times 13 + 1$, $6 \times 7 + 2$ and $6 \times 1 + 3$ is equal to 106.
Case 2. Nina's new amount is 38 and Max's is 53.
From Nina's new amount, Max has 1, 2, 3, 4, 5 or 6 dollars, and Nina has 32, 26, 20, 14, 8 or 2 dollars respectively. This time, $6 \times 8 + 5 = 53$. Hence Max can have as much as 8 dollars.

14. The diagram below shows the net of a solid. Find the number of vertices of this solid.



Solution:
The solid has 7 faces, 1 pentagon, 3 quadrilaterals and 3 triangles. Among them, they have $1 \times 5 + 3 \times 4 + 3 \times 3 = 26$ edges. Since each edge lies on exactly two faces, the solid has $26 \div 2 = 13$ edges. By Euler's Formula, it has $13 + 2 - 7 = 8$ vertices. The solid is shown in the diagram below.



15. C is the midpoint of a semicircular arc with centre O and diameter AB. $OPQR$ is a square with P on OA, Q on the arc AC and R on OC. If the area of $OPQR$ is 8 cm^2, find the area, in cm^2, of the semicircle, taking $\pi = 3.14$.

Solution:
The area of $OPQR$ is equal to half the product of OQ and PR, both of which is equal to the radius of the semicircle. Thus the radius is $\sqrt{2 \times 8} = 4$ cm and the area of the semicircle is $\frac{1}{2} \times 4^2 \times 3.14 = 25.12$ cm^2.