

Team Contest

1. Find the largest number of different positive integers whose sum is 2014.

Solution:

To maximize the number of positive integers, we minimize the positive integers. Thus we start from 1 and go up consecutively. Now $1 + 2 + \dots + 62 = \frac{62 \times 63}{2} = 1953 < 2014$ but $2014 - 1953 = 61$. So we cannot take it up to 62. We have $1 + 2 + \dots + 61 = \frac{61 \times 62}{2} = 1891$ and $2014 - 1891 = 123$. Therefore, we take $1 + 2 + \dots + 61 + 123 = 2014$, using 62 positive integers.

2. Place each of the number 1, 2, 3, 4, 5, 6, 7 and 8 in a different box in the diagram below to make the equation correct. Find one possible solution.

$$\frac{\square}{\square} + \frac{\square}{\square} = \frac{\square}{\square} + \frac{\square}{\square}$$

Solution:

Note that 5 and 7 must be used as numerators since they have no common divisor greater than 1 with any of the other numbers. On the other hand, 3 and 6 may be used as denominators, but only if they are in the same fraction, or both are used as denominators. There are nine solutions overall.

We first consider the special situation where 3 and 6 appear in the same fraction, which is equal to 2 or $\frac{1}{2}$. We cannot have 5 and 7 as numerators on the same side, as the numerator of the sum on that side will be one of $2 \times 5 + 7 = 17$, $2 \times 7 + 5 = 19$, $4 \times 5 + 7 = 27$ and $4 \times 7 + 5 = 33$. This cannot be reproduced on the other side. If the numerator of the other fraction on the same side as 3 and 6 is 7, we have $\frac{3}{6}$ or $\frac{6}{3}$ plus $\frac{7}{1}$, $\frac{7}{2}$, $\frac{7}{4}$ or $\frac{7}{8}$. The only solution is $\frac{6}{3} + \frac{7}{2} = \frac{5}{1} + \frac{4}{8}$. If the numerator of the other fraction on the same side as 3 and 6 is 5, we have $\frac{3}{6}$ or $\frac{6}{3}$ plus $\frac{5}{1}$, $\frac{5}{2}$, $\frac{5}{4}$ or $\frac{5}{8}$. The only solution is $\frac{3}{6} + \frac{5}{1} = \frac{7}{2} + \frac{8}{4}$.

When 3 and 6 are not in the same fraction, the denominators are either 1, 2, 4 and 8, or 3 and 6 and two of 1, 2, 4 and 6. There are seven cases.

Case 1. The denominators are 1, 2, 4 and 8.

Taking 8 as the common denominator, the numerators 3, 5, 6 and 7 are multiplied by 1, 2, 4 and 8 in some order. All possible products are shown in the table below.

×	1	2	4	8
3	3	6	12	24
5	5	10	20	40
6	6	12	24	48
7	7	14	28	56

We must choose four numbers in different rows and different columns so that they form two pairs with equal sum. From the first column, we must take the only even number, namely 6. We cannot take the number 6 from the second column since $20 \neq 56$ and $28 \neq 40$. If we take 14, then $14 - 6 = 8$. However, $24 - 20 = 4$ and $40 - 12 = 28$. So we must take 10 from the second column, and $10 - 6 = 4$. Although $56 - 12 = 44$, we do have $28 - 24 = 4$, yielding $6 + 28 = 10 + 24$. From $\frac{6}{8} + \frac{28}{8} = \frac{10}{8} + \frac{24}{8}$ we have our first solution $\frac{6}{8} + \frac{7}{2} = \frac{5}{4} + \frac{3}{1}$.

Case 2. The denominators are 1, 3, 4 and 6.

Taking 12 as the common denominator, the numerators 2, 5, 7 and 8 are multiplied by 2, 3, 4 and 12 in some order. All possible products are shown in the table below on the left. We do not have four numbers in different rows and different columns which form two pairs with equal sum.

×	2	3	4	12
2	4	6	8	24
5	10	15	20	60
7	14	21	28	84
8	16	24	32	96

×	3	4	8	12
1	3	4	8	12
4	12	16	32	48
5	15	20	40	60
7	21	28	56	84

Case 3. The denominators are 2, 3, 6 and 8.

Taking 24 as the common denominator, the numerators 1, 4, 5 and 7 are multiplied by 3, 4, 8 and 12 in some order. All possible products are shown in the table above on the right. We do not have four numbers in different rows and different columns which form two pairs with equal sum.

Case 4. The denominators are 1, 3, 6 and 8.

Taking 24 as the common denominator, the numerators 2, 4, 5 and 7 are multiplied by 3, 4, 8 and 24 in some order. All possible products are shown in the table below on the left. We have $12+56=20+48$, which yields $\frac{4}{8} + \frac{7}{3} = \frac{5}{6} + \frac{2}{1}$.

×	3	4	8	24
2	6	8	16	48
4	12	16	32	96
5	15	20	40	120
7	21	28	56	168

×	1	2	3	6
4	4	8	12	24
5	5	10	15	30
7	7	14	21	42
8	8	16	24	48

Case 5. The denominators are 1, 2, 3 and 6.

Taking 6 as the common denominator, the numerators 4, 5, 7 and 8 are multiplied by 1, 2, 3 and 6 in some order. All possible products are shown in the table above on the right. We have $7+24=15+16$, which yields $\frac{7}{6} + \frac{4}{1} = \frac{8}{3} + \frac{5}{2}$.

Case 6. The denominators are 2, 3, 4 and 6.

Taking 12 as the common denominator, the numerators 1, 5, 7 and 8 are multiplied by 2, 3, 4 and 6 in some order. All possible products are shown in the table below. We have $2+42=20+24$, which yields $\frac{1}{6} + \frac{7}{2} = \frac{5}{3} + \frac{8}{4}$, as well as $6+28=10+24$, which yields $\frac{1}{2} + \frac{7}{3} = \frac{5}{6} + \frac{8}{4}$.

×	2	3	4	6
1	2	3	4	6
5	10	15	20	30
7	14	21	28	42
8	16	24	32	48

×	2	3	4	6
1	2	3	4	6
5	10	15	20	30
7	14	21	28	56
8	16	24	32	48

Case 7. The denominators are 3, 4, 6 and 8.

Taking 24 as the common denominator, the numerators 1, 2, 5 and 7 are multiplied by 3, 4, 6 and 8 in some order. All possible products are shown in the table below. We have $4+42=6+40$, which yields $\frac{1}{6} + \frac{7}{4} = \frac{2}{8} + \frac{5}{3}$, as well as $6+30=8+28$, which yields $\frac{2}{8} + \frac{5}{4} = \frac{1}{3} + \frac{8}{4}$.

×	3	4	6	8
1	3	4	6	8
2	6	8	12	16
5	15	20	30	40
7	21	28	42	56

×	3	4	6	8
1	3	4	6	8
2	6	8	12	16
5	15	20	30	40
7	21	28	42	56

3. The number 12345678901112...484950 is obtained by writing the integers from 1 to 50 inclusive one after the other in their natural order. Some digits are removed so that the remaining ones have sum 200. Find the first ten digits from the left of the largest possible number obtained this way.

Solution:

The sum of the units digits of the numbers from 1 to 50 inclusive is $5 \times (1 + 2 + \dots + 9) = 225$. The sum of the tens digits is $10 \times (1 + 2 + 3 + 4) + 5 = 105$. Hence the total is $225 + 105 = 330$. We must eliminate a number of digits with sum 130. To maximize the remaining number, we maximize its number of digits. This is achieved by eliminating large digits. There are five 7s, five 8s and five 9s among the units digits of the numbers from 1 to 50, with sum $5 \times (7 + 8 + 9) = 120$. Thus two more digits with sum 6 must go. They are either 4 and 6 or 5 and 5. The best we can do is to eliminate the first 4 and the last 6. Thus the first ten digits of the maximum value of the remaining number are 1235610111.

4. What is the sum of the units digit of $1^2 + 2^2 + 3^3 + \dots + 1007^{1007}$ and the units digit of $1008^{1008} + 1009^{1009} + 1010^{1010} + \dots + 2014^{2014}$?

Solution:

Since we are only interested in the units digit, we can reduce all base numbers to their units digits. Any power of 0 ends in 0. Any powers of 1 ends in 1. Any powers of 5 ends in 5. Any power of 6 ends in 6. The powers of 2 comes in a 4-cycle 2, 4, 8 and 6. The 2nd power ends in 4, the 12th power ends in 6, so that the relevant powers of 2 comes in a 2-cycle 4 and 6. Similarly, relevant powers of 3 comes in a 2-cycle 7 and 3, the relevant powers of 4 always ends in 6, the relevant powers of 7 comes in a 2-cycle 3 and 7, the relevant powers of 8 comes in a 2-cycle 6 and 4, and the relevant powers of 9 always ends in 9. It follows that the units digit of the sum of every 20 terms is 4, so that the units digit of the sum of the first 1000 terms is 0. The sum of the units digits of the next 7 terms is $1 + 4 + 7 + 6 + 5 + 6 + 3 = 32$. Hence the units digit of $1^2 + 2^2 + 3^3 + \dots + 1007^{1007}$ is 2. The sum of the units digits of the next 1000 term is 0, and the sum of the next 7 terms is $6 + 9 + 0 + 1 + 6 + 3 + 6 = 31$. Hence the units digit of $1008^{1008} + 1009^{1009} + 1010^{1010} + \dots + 2014^{2014}$ is 1. The desired sum is $2 + 1 = 3$.

5. In a positive integer, each digit after the leftmost one is larger than the digit to its left. Its square, which is a six-digit number, also has the same property. Find the original positive integer.

Solution:

The units digit of the square is at least 6. Since it cannot be 7 or 8, it must be 6 or 9. If it is 6, then the square must be 123456. However, this number is divisible by 3 but not by 9. Hence it is not a square. It follows that the units digit of the square ends in 9 so that the units digit of the original number is 3 or 7.

If it is 3, then the original number must be 123, but 123^2 is not even a six-digit number. Hence it must be 7. The square is at least 123459 and at most 456789. Since $351^2 < 123459 < 352^2$ and $675^2 < 456789 < 676^2$, the original number is at least 352 and at most 675. Hence it is one of 357, 367, 457, 467 and 567. Now $357^2 = 127449$, $367^2 = 134689$, $457^2 = 208849$, $467^2 = 218089$ and $567^2 = 321489$. The only number with all the desired properties is 367.

6. The list 9, 18, 27, 36, ... consists of all positive integers whose digit sum is 9 in increasing order. What is the 2014th number in this list?

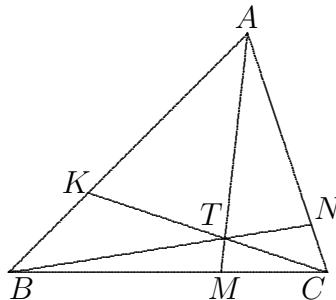
Solution:

This list contains exactly 1 number with at most 1 digit, 10 numbers with at most 2 digits, and so on. To reach 2014 numbers, we try to determine how many numbers have at most 6 digits. To do this, consider an array of 9 zeros and 5 ones, such as 00100011010100. Interpret the zeros as units and the ones as separators. This array corresponds to the number 230112. Conversely, to convert the number 005103 into such an array, we have 0, 0, 5, 1, 0 and 3 zeros between adjacent ones, yielding 11000001011000. Thus we have a one-to-one correspondence. The number of arrays is the number of ways of choosing 5 of 14 slots to place the ones. This can be done in $14 \times 13 \times 12 \times 11 \times 10$ ways if the ones are distinct. Since they are identical, we must divide the above total by $5 \times 4 \times 3 \times 2 \times 1$. The quotient is 2002. This means that the 2014th number in the list is the 12th seven-digit number in the list. The first twelve such numbers are 1000008, 1000017, 1000026, 1000035, 1000044, 1000053, 1000062, 1000071, 1000080, 1000107, 1000116 and 1000125. The last one is the number we seek.

7. K is a point on the side AB and M is a point on the side BC of triangle ABC such that $\frac{AK}{KB} = \frac{BM}{MC} = 2$. AM intersects CK at the point T , and the extension of BT intersects CA at the point N . If the area of triangle AKT is 20 cm^2 , find the area, in cm^2 , of triangle ANT .

Solution:

The area of triangle BKT is equal to $\frac{1}{2} \times 20 = 10 \text{ cm}^2$ since it is $\frac{BK}{AK}$ times the area of AKT . The area of triangle CAT is equal to $\frac{1}{2} \times (20 + 10) = 15 \text{ cm}^2$ since it is $\frac{CM}{BM}$ times the area of BAT . The area of triangle BCT is equal to $\frac{1}{2} \times 15 = 7\frac{1}{2} \text{ cm}^2$ since it is $\frac{BK}{AK}$ times the area of CAT . It follows that $\frac{CN}{AN} = \frac{7\frac{1}{2}}{30} = \frac{1}{4}$ since it is the ratio of the area of BCT to the area of BAT . Hence the area of triangle ANT is equal to $\frac{4}{1+4} \times 15 = 12 \text{ cm}^2$ since it is $\frac{AN}{CA}$ times the area of CAT .



8. How many five-digit multiples of 495 are there such that all five digits are different?

Solution:

Since $495 = 5 \times 9 \times 11$, the units digit is 0 or 5, the digit sum is a multiple of 9 and the alternate digit sum is a multiple of 11. We consider two cases.

Case A. The units digit is 0.

The digit sum is at least $1+2+3+4=10$ and at most $9+8+7+6=30$. Hence it is either 18 or 27. Suppose it is 27. Then the alternate digit sum must also be odd, and is either 11 or -11 . This means that the sum of the two digits on the positive side must be $\frac{27+11}{2} = 19$. However, the sum of any two digits is at most $9+8=17$. Hence the digit sum must be 18, and the alternate digit sum must be 0. It follows that the sum of the ten-thousands digit and the hundreds digit is 9, and so is the sum of the thousands digit and the tens digit. We have $9=1+8=2+7=3+6=4+5$. The two pairs of digits can be chosen in $4 \times 3 = 12$ ways, and the four digits may be arranged in $2 \times 2 = 4$ ways. This yields a total of $12 \times 4 = 48$ numbers.

Case B. The units digit is 5.

The digit sum is still 18 or 27. Suppose it is 27. Then the alternate digit sum is either 11 or -11 . Consider the four-digit number obtained by deleting the digit 5. Its digit sum is 22 and its alternate digit sum is either 6 or -16 . It cannot be -16 as otherwise the sum of the thousands digit and the tens digit must be $\frac{22+16}{2} = 38$, which is impossible. Hence the alternate digit sum is 6. The sum of the thousands digit and the tens digit is $\frac{22-6}{2} = 8$ and the sum of the hundreds digit and the units digit is $22 - 8 = 14$. We must have $14=6+8$ and $8=1+7$. This yields 4 numbers. If the digit sum of the five-digit number is 18, then the alternate digit sum must be 0. Consider the four-digit number obtained by deleting the digit 5. Its digit sum is 13 and its alternate digit sum is either -5 . Hence the sum of the hundreds digit and the units digit must be $\frac{13+5}{2} = 9$, and the sum of the thousands digit and the tens digit is $13 - 9 = 4$. If we have $4=1+3$, then $9=0+9=2+7$. This yields 8 numbers. If we have $4=0+4$, then $9=1+8=2+7=3+6$. This yields only 6 numbers since the thousands digit cannot be 0.

The grand total is $48+4+8+6=66$.

9. In the 3×3 table below, each letter stands for a different one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9. The sum of the three numbers in each row is 15, and the sum of the three numbers in each column is 15.

A	B	C
D	E	F
G	H	I

Peter guesses that $A=2$ and $C=9$. Tom guesses that $E=5$ and $F=8$. Sam guesses that $G=3$ and $H=7$. One of them guesses right both times, another guesses wrong both times while the third one guesses right once and wrong once. Find the product of the B, D, F and H.

Solution:

There are eight ways of forming 15, namely $1+5+9$, $1+6+8$, $2+4+9$, $2+5+8$, $2+6+7$, $3+4+8$, $3+5+7$ and $4+5+6$. We consider three cases.

Case 1. Peter is right both times.

Then $B=4$. The third column can only be $1+5+9$, so that the first column must be $2+6+7$. Hence Tom and Sam are wrong both times. This is impossible.

Case 2. Tom is right both times.

Then $D=2$, and $A=2$ is wrong. Now 3 cannot appear in the first column and 9 cannot appear in the third column. Hence $H=7$ is right, but neither $2+6+7$ nor $3+5+7$ is available for the third row. This is impossible.

Case 3. Sam is right both times.

Then $I=5$. The first column must be $3+4+8$ and the second column $2+6+7$, so that the third column is $1+5+9$. Now $A=2$, $E=5$ and $F=8$ are all wrong, so that $C=9$ is right. This leads to the unique solution shown below.

4	2	9
8	6	1
3	7	5

10. On a table is a $5 \times 5 \times 5$ block formed from 125 unit cubes. Each minute, all unit cubes with at least three exposed faces are removed, so that some faces formerly hidden are now exposed. How many unit cubes remain on the table after 5 minutes?

Solution:

Consider a $1 \times 5 \times 5$ block instead, with a 5×5 face on the table. In the diagram below, the number on each square denotes the minute during which the corresponding unit cube is removed. We now consider the $5 \times 5 \times 5$ block. No cube in the top layer survives. Only the cube corresponding to the square labelled 5 survives in the second layer. Only the cubes corresponding to the squares labelled 4 or 5 survive in the third layer. Only the unit cubes corresponding to the squares labelled 1 and 2 are removed from the fourth layer, and only the cubes corresponding to the squares labelled 1 are removed from the fifth layer. Thus total number of surviving cubes is $0+1+5+13+21=40$.

1	2	3	2	1
2	3	4	3	2
3	4	5	4	3
2	3	4	3	2
1	2	3	2	1