Po Leung Kuk 21st Primary Mathematics World Contest
Individual Contest

1. Let $A = 2^{2018} + 3^{2018}$. What is the unit digit of A?

【Solution】

Observe that $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$ and so on. Therefore, the units digit will be in the pattern of 2, 4, 8, 6, 2, 4, 8, 6 and so on. It is periodic with period 4. Since $2018 = 4 \times 504 + 2$, the units digit of $2^{2018} = 2^{4\times 504} \times 2^2$ is 4.

Similarly, observe that $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$ and so on. Therefore, the units digit will be in the pattern of 3, 9, 7, 1, 3, 9, 7, 1 and so on. It is periodic with period 4. Since $2018 = 4 \times 504 + 2$, the units digit of $3^{2018} = 3^{4\times 504} \times 3^2$ is 9.

For get the units digit of A, we just get the sum of both results above which is $4 + 9 = 13$, therefore, the unit digit of $A = 2^{2018} + 3^{2018}$ is 3.

Answer：3

2. There are 4 children standing in a line.

The distance between Amy and Cindy is 12 m longer than the distance between Cindy and Don. The distance between Amy and Billy is 3 m shorter than the distance between Billy and Don. What is the distance, in m, between Billy and Cindy?

【Solution】

Let the distance between A and C be $AB + BC$ and the distance between B and D be $BC + CD$.

From the given conditions, we have

$$AC - CD = AB + BC - CD = 12 \qquad (1)$$

$$BD - AB = BC + CD - AB = 3 \qquad (2)$$

Adding (1) and (2), we get $2BC = 15$, thus, $BC = \frac{15}{2} = 7\frac{1}{2} = 7.5\,\text{m}$.

Answer： $\frac{15}{2} = 7\frac{1}{2} = 7.5\,\text{m}$

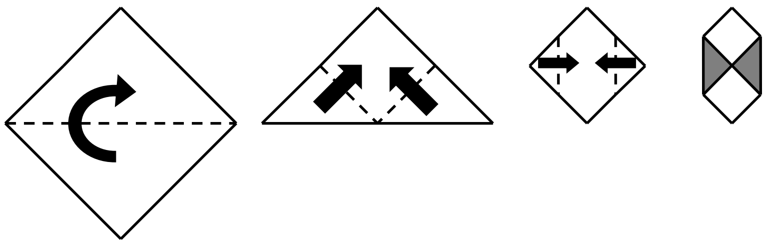
3. We put some balls into 10 boxes such that each box has a different number of balls with the following conditions on the number of balls in each box:
   (1) cannot be less than 11;
   (2) cannot be 13;
   (3) cannot be a multiple of 5.
   What is the least total number of balls needed to satisfy these three conditions?

【Solution】

In order to minimize the total number of balls to be placed in the 10 boxes to satisfy the conditions, we can make the number of balls placed in each box to be 11, 12, 14, 16, 17, 18, 19, 21, 22 and 23. Therefore, the least total number of balls needed to satisfy all three conditions is $11+12+14+16+17+18+19+21+22+23=173$ balls.

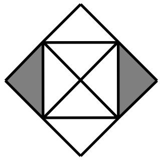
Answer：173 balls

4. A square piece of paper of area 100 cm^2, is folded in half along the dotted line as shown below. Every fold thereafter is exactly through the midpoint of the line segment. What is the area, in cm^2, of the shaded region in the last figure?



【Solution #1】

It is easy to see that the area of second figure from the left is $100 \div 2 = 50\,\text{cm}^2$ and the area of the third figure from the left is $50 \div 2 = 25\,\text{cm}^2$. Now, we can see that the shaded area of the last figure is just two-eighths (or one-fourth) of the total figure, therefore, the area of the shaded region in the rightmost figure is $25 \div 4 = 6\frac{1}{4} = 6.25\ \text{cm}^2$.



【Solution #2】

It is easy to see the position of the rightmost figure in the original square figure as shown in Figure 1.

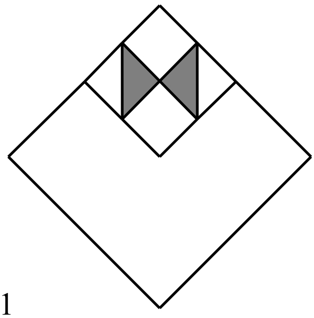


Figure 1

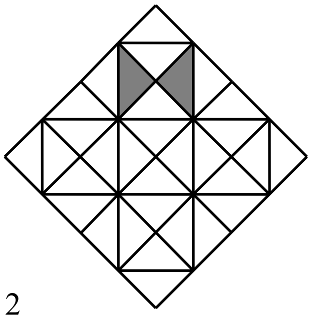


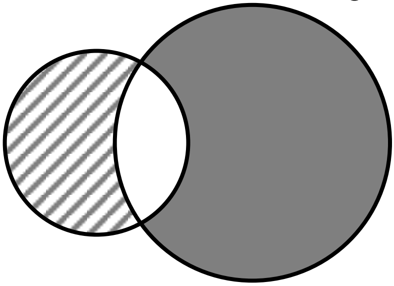
Figure 2

Now, the original square paper can be cut into 32 equal triangles as shown in Figure 2. Since 2 triangles are shaded, therefore, the area of the shaded region is $100 \times \frac{2}{32} = \frac{100}{16} = \frac{25}{4} = 6\frac{1}{4} = 6.25\,\text{cm}^2$.

Answer： $6\frac{1}{4} = 6.25\ \text{cm}^2$

5. In the diagram below, $\frac{4}{7}$ of the small circle is shaded, and $\frac{5}{6}$ of the large circle is shaded. What is the ratio of the shaded area

of the small circle to the shaded area of the large circle?

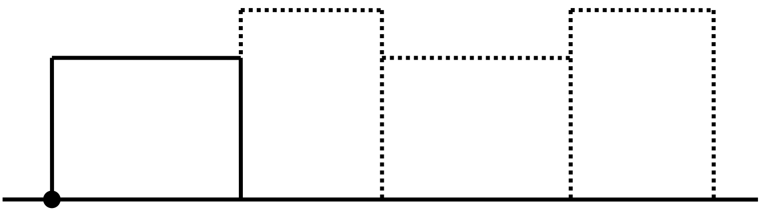


【Solution】

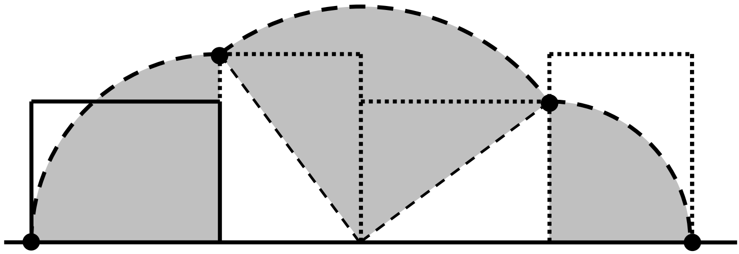
It is known that the unshaded portion of the small circle is $1-\frac{4}{7}=\frac{3}{7}$, while the unshaded portion of the large circle is $1-\frac{5}{6}=\frac{1}{6}$. From this, we can say that the area of the intersection of the two circles is 3 units, the area of the small circle is 7 units and the area of the large circle is 18 units. From this, we can deduce that the area of the shaded portion of the small circle is 7 - 3=4 units, while the area of the shaded portion of the large circle is 18 - 3=15 units. Therefore, the ratio of the shaded area of the small circle to the shaded area of the large circle is 4 : 15.

Answer : 4 : 15

6. A rectangle of length 4 cm and width 3 cm is in position **I**. Rotate the rectangle clockwise about point *B* to position **II**. Then rotate the rectangle clockwise about point *C* to position **III**, and rotate the rectangle clockwise about point *D* to position **IV**. Points *A*, *B*, *C*, *D* and *E* lie on a straight line. How long, in cm, is the distance that point *A* travels? (Let $\pi=3.14$)



【Solution】



Draw the trajectory paths of point A as shown by the black dotted lines in the figure above. It can be seen that from position **I** to position **II**, the trajectory of point A is a quarter circle with radius 4 cm. Similarly, from position **II** to position **III**, the trajectory of point A is again a quarter circle with radius 5 cm. Lastly, from position **III** to position **IV**, the trajectory of point A is again a quarter circle with radius 3 cm. Therefore, the total distance that point A travelled is $\frac{1}{4}(4+5+3)\times 2\times 3.14 = 18.84$ cm.

Answer：18.84 cm

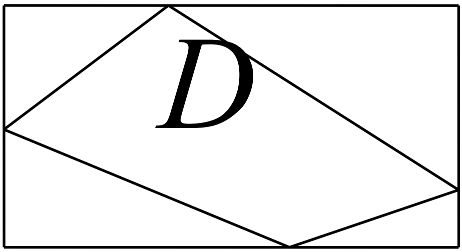
7. There are 4 different story books, 3 different comic books and 2 different language books to be arranged in a row. The story books are required to be put together, the comic books are required to be put together and the language books are required to be put together. How many different arrangements of the nine books are possible?

【Solution】

The number of ways that we can arrange 4 different story books in a row is $4\times 3\times 2\times 1 = 24$ ways, the number of ways we can arrange 3 different comic books is $3\times 2\times 1 = 6$ ways, and the number of ways we can arrange 2 different language books is $2\times1=2$ ways, and the number of ways we can arrange the order of these books in a row is $3\times2\times1=6$ ways. Therefore, there is a total number of $24\times6\times2\times6=1728$ possible arrangements for the nine books.

Answer：1728 ways

8. The rectangle $ABCD$ has area 30 cm^2. Points E, F, G and H lie on the line segments AB, BC, CD and DA respectively. Suppose $AH=FB+1$ cm and $AE=DG+2$ cm, find the area, in cm^2, of quadrilateral $EFGH$.



【Solution #1】

Let G and E be points such that GL and EN are both parallel to BC, while H and F be points such that HN and FL are both parallel to AB. Segment HN have intersection points on GL and EN at points K and N, respectively, while segment FL have intersection points on GL and EN at points L and M, respectively, as shown in the figure below.

Since $DHKG$, $GLFC$, $BFME$ and $AENH$ are all rectangles, and let [*]

denote the area of a polygon *, then, $[\triangle DHG] = [\triangle GHK]$, $[\triangle GCF] = [\triangle GFL]$, $[\triangle BEF] = [\triangle MEF]$ and $[\triangle AHE] = [\triangle NHE]$. Therefore,

$$
\begin{aligned}
&[\square EFGH] \\
&= [\triangle HKG] + [\triangle GHF] + [\triangle MEF] + [\triangle NHE] - [\square KLMN] \\
&= \frac{1}{2}[\square DHKG] + \frac{1}{2}[\square GLFC] + \frac{1}{2}[\square MEBF] \\
&\quad + \frac{1}{2}[\square AENH] - [\square KLMN] \\
&= \frac{1}{2}[\square ABCD] + \frac{1}{2}[\square KLMN] - [\square KLMN] \\
&= \frac{1}{2}[\square ABCD] - \frac{1}{2}[\square KLMN] \\
&= \frac{1}{2}(30 - 2\times 1) \\
&= 14 \text{ cm}^2
\end{aligned}
$$

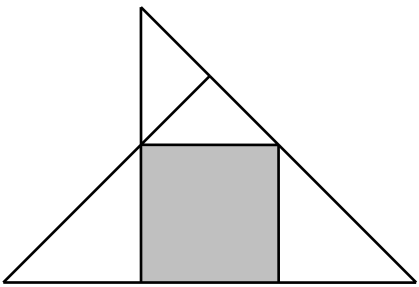
【Solution #2】

Let $FB = x$ cm, $CF = y$ cm, $DG = s$ cm and $GC = t$ cm, then $AH = x+1$ cm, $DH = y-1$ cm, $AE = s+2$ cm and $EB = t-2$ cm. Therefore,

$$
\begin{aligned}
&[\square EFGH]\\
&=[ABCD]-[\triangle DHG]-[\triangle GFC]-[\triangle BEF]-[\triangle AHE]\\
&=(s+t)(x+y)-\frac{1}{2}s(y-1)-\frac{1}{2}ty-\frac{1}{2}x(t-2)-\frac{1}{2}(x+1)(s+2)\\
&=(s+t)(x+y)-\frac{1}{2}(sy-s+ty+xt-2x+xs+2x+s+2)\\
&=\frac{1}{2}(s+t)(x+y)-1\\
&=15-1\\
&=14
\end{aligned}
$$

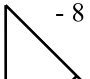
Answer：14 cm^2

9. In the diagram below, the two right-angled isosceles triangles, $\triangle ABC$ and $\triangle DEC$, are put together such that D, A, C are collinear and B, E, C are collinear. AB and DE intersect at point F. Points G and H lie on the line segments DC and BC respectively. If the shaded region $EFGH$ is a square, then what is the ratio of the area of $\triangle ABC$ to the area of $\triangle DEC$?



【Solution #1】

We divide the original figure with equal triangles as shown below. Notice that there are 9 small triangles in $\triangle ABC$, while there are 8 small triangles in $\triangle DEC$, thus, the ratio of the area of $\triangle ABC$ to the



area of $\triangle DEC$ is 9 ∶ 8.

【Solution #2】

If the side length of the shaded square is 1, we can say that the length of the hypothenuse of $\triangle ABC$ is 3, so it's square is 9. Also, notice that the square of the hypotenuse of isosceles right triangle GHC is $1^2+1^2=2$. Since $DC=2GC$, then the square of the hypotenuse of the isosceles right triangle DEC is $DC^2=4GC^2=8$. Since the ratio of the areas of the isosceles right triangles is just the ratio of the squares of its corresponding side lengths, therefore, the ratio of the area of $\triangle ABC$ and $\triangle DEC$ is equal to 9:8.

Answer ∶ 9 ∶ 8

10. A 3×3 grid should be filled with a number in each square such that the sum of the three numbers in each row, column and the two main diagonals are equal. The partially completed grid is shown below. What is the value of *a*?

a	16	20
2		

【Solution】

Let the numbers in the remaining five empty squares be *b*, *c*, *d*, *e* and *f*, as shown in the figure.

a	16	20
2	c	e

b	d	f

Since the sum of the three numbers in each row, column and the two main diagonals are equal, therefore, $16+20=2+b$, so $b=34$.

Since $b+20=16+d$, therefore $d=34+20-16=38$;

Since $b+d=20+e$, therefore $e=34+38-20=52$;

Since $a+2+b=16+c+d$, therefore,

$c=a+2+34-16-38=a-18$;

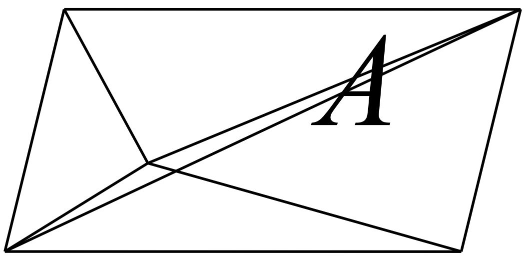
and lastly $a+c=20+e$ therefore, $a+a-18=20+52$, thus,

$a=45$。Now we know that $c=27$ and $f=9$ to complete the grid.

45	16	20
2	27	52
34	38	9

Answer：45

11. In the diagram below, point *P* is inside parallelogram *ABCD*. If the area of $\triangle ABP$ and $\triangle BPC$ is 73 cm^2 and 100 cm^2, respectively, find the area, in cm^2, of $\triangle BPD$.



【Solution】

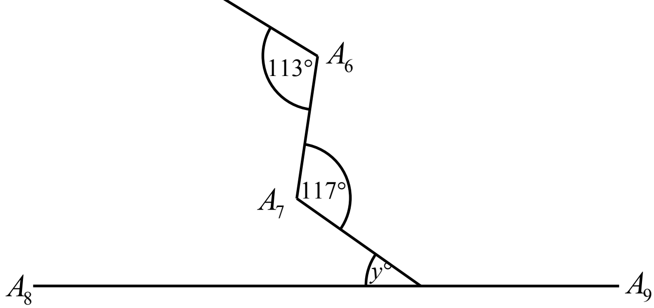
Since *ABCD* is a parallelogram and [*] denotes the area of a polygon *, then

$$[\triangle BPC]+[\triangle APD]=\frac{1}{2}[\square ABCD]$$
$$=[\triangle ABP]+[\triangle APD]+[\triangle BPD]$$

Therefore, $[\triangle BPD]=[\triangle BPC]-[\triangle ABP]=100-73=27\text{ cm}^2$.

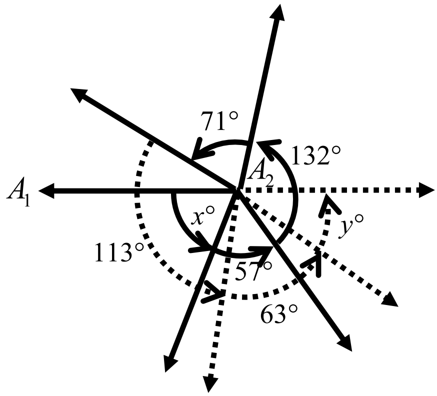
Answer：27 cm^2

12. In the figure below, $\overline{A_1A_2}$ isparallel to $\overline{A_8A_9}$. Find the value of $x+y$.



【Solution #1】

Add arrows to A_1 to show direction. Notice that it is rotating counterclockwise in the manner shown in the figure below.



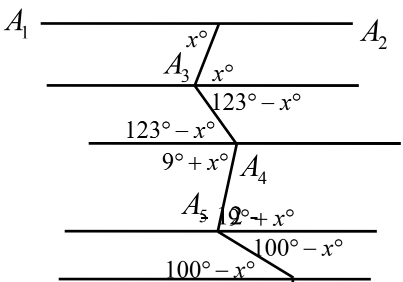
Now, we can see that the figure rotates a full circle and a half, that is $540°$. So,

$$540° = x° + 57° + 132° + 71° + 113° + 63° + y°$$

Simplifying, we have $x + y = 104$.

【Solution #2】

Draw lines in each on A_3, A_4, A_5, A_6 and A_7 that are parallel to $\overline{A_1A_2}$. From this, the equal angles of these parallel lines are seen as shown in the following figure.



Then, $104° - x° = y°$, therefore, $x + y = 104$.

Answer：104

13. The fraction $\dfrac{221}{210}$ is obtained as a sum of three positive fractions each less than 1 with single digit denominators. Find the largest (greatest) of these fractions in simplest form.

【Solution】

Observe that $210 = 2\times3\times5\times7$, and since the denominators of these three fractions are all single digits, so the denominators should be 5, 6 and 7. Let $\dfrac{221}{210}=\dfrac{a}{5}+\dfrac{b}{6}+\dfrac{c}{7}$, where b and 6 are relatively prime, then $\dfrac{221}{210}=\dfrac{42a}{210}+\dfrac{35b}{210}+\dfrac{30c}{210}$, now, we have $221 = 42a+35b+30c$. Observe that $42\times5 < 221 < 42\times6$, we can deduce that $a<6$, and since $42a$ and $30c$ are even numbers, therefore must be an odd number, and the units digit of of $42a$ is 6, therefore, $a=3$, so $35b+30c=95$, then we can deduce that $b=1$ and $c=2$, therefore, the three fractions are $\dfrac{3}{5}$, $\dfrac{1}{6}$ and $\dfrac{2}{7}$, and the largest is $\dfrac{3}{5}$.

Answer：$\dfrac{3}{5}$

14. Sixteen husband and wife couples attend a party. This party has the condition that every husband will shake hands with all the other guests except his own wife and wives do not shake hands

with other wives. Find the total number of handshakes between the thirty-two people at the party.

【Solution #1】

If all sixteen couples shook hands with each other without any restrictions, then there is a total of $\frac{32\times31}{2}=496$ handshakes. Now, we must deduct the handshakes between wives since each wife just shook hands only once and this is $\frac{16\times15}{2}=120$ handshakes and we must also deduct the handshakes between each couple which is 16 times, therefore, the total number of handshakes between the thirty-two people at the party is $496-120-16=360$ handshakes.

【Solution #2】

The husbands each shook hands for a total of $\frac{16\times15}{2}=120$ times, and the husband and wife of the different pairs each shook hands for a total of $16\times15=240$ times, so total number of handshakes between the people at the party is $120+240=360$ handshakes.

Answer：360 handshakes

15. In a hockey tournament, a win is worth 2 points, a draw is worth 1 point and a loss is worth 0 points. There are ten teams, and each team plays every other team exactly once. What is the maximum number of teams with at least 12 points at the end of the tournament?

【Solution】

Since each team plays every other team exactly once, then there is a total of $\frac{10\times9}{2}=45$ games played. Moreover, since in each game played, 2 points is at stake, then there is a total of $45\times2=90$ points given out. Since $90=12\times7+6$, therefore at least seven teams have at least 12 points. This scenario to possible to happen. One scenario is when the results of the match between the seven teams are all draws, and all these seven teams win their games against the other three teams, then each of these seven teams scores $6+2\times3=12$ points.

Answer：7 teams