Key to Po Leung Kuk 21st Primary Mathematics World Contest
Team Contest

1. Let x, y, m and n be positive integers such that
   (1) $x > y$, $x + y = 7$, $xy = 12$,
   (2) $m > n$, $m + n = 13$ and $m^2 + n^2 = 97$.
   Let $A = x - y$ and $B = m - n$. Find the value of $A + B$.

【Solution #1】

From the given information, x, y are positive integers and $x > y$, $xy = 12$, so $(x, y) = (12,1)$, $(6,2)$ or $(4,3)$. Again, we have $x + y = 7$, then $x = 4$, $y = 3$ and $A = x - y = 1$.
Since $m^2 + n^2 = 97$, so $m \le 9$, but we also have $m > n$ and $m + n = 13$, then $m \ge 7$.
Because m and n are positive integers, this implies that among the three possible values of m, only $m = 9$, $n = 4$ satisfied the given condition, it follows that $B = m - n = 5$. Hence, $A + B = 6$.

【Solution #2】

Since x, y are positive integers and $x > y$, $xy = 12$, so $(x, y) = (12,1)$, $(6,2)$ or $(4,3)$.
We also know that $x + y = 7$, then $x = 4$, $y = 3$ and $A = x - y = 1$.
But m, n are positive integers and $m + n = 13$, $m^2 + n^2 = 97$, then from $m = 13 - n$ we obtain $97 = (13 - n)^2 + n^2 = 169 - 26n + 2n^2$, which is $n^2 - 13n + 36 = 0$, so apply factoring $(n - 9)(n - 4) = 0$, from $m > n$ we know that $m = 9$, $n = 4$ then $B = m - n = 5$.
Thus, $A + B = 6$.

Answer: 6

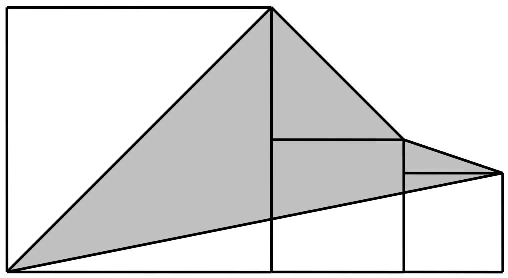
2. Three pirates were gambling together. Initially, the money they each had was in the ratio 7:6:5. After the game, the money that each of them had was in the ratio 6:5:4 (in the same order). If one of them won 12 dollars, how much money, in dollars, did they have to gamble with in total?

【Solution】

We know that at the beginning, the money of each pirate is in the ratio of 7 : 6 : 5, so the total amount of money of three pirates must be a multiple of 18 $(= 7 + 6 + 5)$.
While at the end of the game, their money ratio is 6: 5 : 4, so the total amount of three pirates' money is a multiple of 15 $(= 6 + 5 + 4)$.
Since $\text{GCF}(18,\ 15) = 90$, then the total amount of money of these three pirates is a multiple of 90, so the ratio of money they have at the beginning is equivalent as 35:30:25, and the ratio of the money they have at the end can be expanded as 36:30:24. It is known that one of them wins \$12, from the result of the expansion of ratio, we can be conclude only the first pirate has won in gambling and the amount of money he/she won is $\frac{1}{90}$ the total amount of three people.
Therefore, the three pirates have a total of $12 \times 90 = 1080$ dollars.

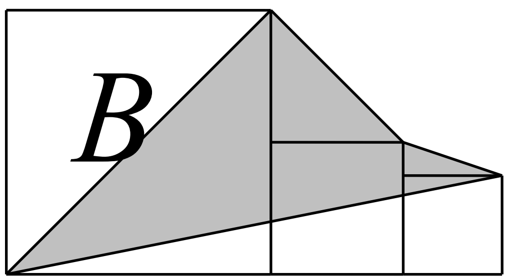
Answer: \$1080

3. The following figure consists of three squares of sides 8 cm, 4 cm and 3 cm respectively. Find the area, in cm^2, of the shaded region.



【Solution 】

Mark each point in the given figure.



We know that area of $\triangle CGD$ is $\frac{1}{2}\times 4\times(8-4)=\frac{1}{2}\times 4\times 4=8\,\text{cm}^2$,

while the area of $\triangle DHE$ is $\frac{1}{2}\times 3\times(4-3)=\frac{1}{2}\times 3\times 1=1\frac{1}{2}\,\text{cm}^2$.

It follows the area of the entire figure is $8\times 8+4\times 4+3\times 3+8+1\frac{1}{2}=98\frac{1}{2}\,\text{cm}^2$.

But the area of $\triangle ABC$ is $\frac{1}{2}\times 8\times 8=32\,\text{cm}^2$

and the area of $\triangle AEF$ is $\frac{1}{2}\times 3\times(8+4+3)=22\frac{1}{2}\,\text{cm}^2$.

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Thus, the area of the total shaded region is $98\frac{1}{2}-32-22\frac{1}{2}=44\,\text{cm}^2$.

Answer: 44 cm^2

4. The sum of the first 2018 terms of the sequence 1, –2, 3, 4, –5, 6, 7, –8, 9, 10, –11, 12, … is n. The sum of the first 2018 terms of the sequence 1, 2, –3, 4, 5, –6, 7, 8, –9, 10, 11, –12, … is m. What is the value of $m-n$?

【Solution #1】

A

$$
\begin{aligned}
& m-n \\
&=\{[1+2+(-3)]+[4+5+(-6)]+\cdots+[2014+2015+(-2016)] \\
&\quad +2017+2018\}-\{[1+(-2)+3]+[4+(-5)+6]+\cdots \\
&\quad +[2014+(-2015)+2016]+2017+(-2018)\} \\
&=\{[1+2+(-3)]-[1+(-2)+3]\}+\{[4+5+(-6)]-[4+(-5)+6]\} \\
&\quad +\cdots+\{[2014+2015+(-2016)]-[2014+(-2015)+2016]\} \\
&\quad +\{[2017+2018]-[2017+(-2018)]\} \\
&=\underbrace{(-2)+(-2)+\cdots+(-2)}_{\text{(672 terms)}}+4036 \\
&=4036-672\times 2 \\
&=2692
\end{aligned}
$$

【Solution #2】

$$
\begin{aligned}
n &=1+(-2)+3+4+(-5)+6+\cdots+2016+2017+(-2018) \\
&=1+2+3+4+5+\cdots+2018-2\times(2+5+8+\cdots+2018) \\
&=\frac{(1+2018)\times 2018}{2}-(2+2018)\times 673
\end{aligned}
$$

$$
\begin{aligned}
m &=1+2+(-3)+4+5+(-6)+\cdots+(-2016)+2017+2018 \\
&=1+2+3+4+5+\cdots+2018-2\times(3+6+9+\cdots+2016) \\
&=\frac{(1+2018)\times 2018}{2}-(3+2016)\times 672
\end{aligned}
$$

Hence, $m-n=(2+2018)\times 673-(3+2016)\times 672=2692.$

Answer: 2692

5. $S(n)$ is defined as "the sum of the digits of n" where n is a positive integer. For example, $S(3)=3$ and $S(32)=3+2=5$. Find one positive integer n such that $n+S(n)=2018$.

【Solution 】

From the given information, $n=2018-S(n)$ and $S(n)\geq 0,$ so $n\leq 2018$.

It follows that $S(n)\leq 1+9+9+9=28,$ then $n\geq 2018-28=1990$.

When the first digit of n is 1, ones digit of n is an even number, then $S(n)$ will be odd, and it is impossible $n+S(n)$ equal to 2018. Suppose ones digit of n is an odd number, then $S(n)$ will be even, and it is also impossible $n+S(n)$ equal to 2018. Hence, the first digit of n is not equal to 1.

When the first digit of n is 2, assume $n=\overline{2abc}$, then $\overline{2abc}+2+a+b+c=2018,$ which is equivalent as $101a+11b+2c=16$. This implies a must equal to 0, then $11b+2c=16$ it follows $b\leq 1$. Since both $2c$ and 16 are even number, so $11b$ must also be even number, then $b=0$, and so $c=8$. Therefore, only $n=2008$ will satisfy the condition of the problem.

Answer: 2008

6. There are 3 fractions in simplest form, $\frac{a}{15}$, $\frac{b}{21}$ and $\frac{c}{35}$ where *a*, *b* and *c* are single digit positive integers. If the sum of these three fractions is an integer, what is the minimum value of $a+b+c$? (Note that *a*, *b* and *c* are not necessarily distinct.)

【Solution 1】

We know that $\frac{a}{15}+\frac{b}{21}+\frac{c}{35}\le\frac{9}{15}+\frac{9}{21}+\frac{9}{35}=\frac{135}{105}$,

then $\frac{a}{15}+\frac{b}{21}+\frac{c}{35}=1$, which is equivalent as $7a+5b+3c=105$.

Since $\frac{a}{15}$ is a simplest fraction, so *a* cannot be 3, 5, 6 or 9.

Also $\frac{b}{21}$ is a simplest fraction, so *b* cannot be 3, 6, 7 or 9.

Similar, $\frac{c}{35}$ is a simplest fraction, so *c* cannot be 5 or 7.

But $7a=105-(5b+3c)\ge105-(5\times9+3\times9)=33$, hence $a\ge5$.

Case 1: When $a=7$, $5b+3c=56$. But $5b+3\times9\ge5b+3c=56$, that is; $5b\ge56-27=29$, then $b\ge6$, so b can only be equal to 8, but at this time *c* is not an integer, hence no solution.

Case 2: When $a=8$, $5b+3c=49$. Since $5b+3\times9\ge5b+3c=49$, that is; $5b\ge49-27=22$, then $b\ge5$, so b can only be equal to 5 or 8.

If $b=5$, $c=8$, and we have $a+b+c=21$.

If $b=8$, $c=3$, and we have $a+b+c=19$.

Thus, the minimum value of $a+b+c$ is 19.

【Solution 2】

We know that $\frac{a}{15}+\frac{b}{21}+\frac{c}{35}\le\frac{9}{15}+\frac{9}{21}+\frac{9}{35}=\frac{135}{105}$, then $\frac{a}{15}+\frac{b}{21}+\frac{c}{35}=1$, which is equivalent as $7a+5b+3c=105$.

Rewrite the above equation as $4a+2b=105-3(a+b+c)$. We know that finding the minimum value of $a+b+c$ is equivalent as to find the maximum value of $4a+2b$:

Case 1: When $a=b=9$, $4a+2b=54$ but $7a+5b=108>105$, a contradiction!

Case 2: When $a=9$, $b=8$, $4a+2b=52$ but $3c=105-63-40=2$, then *c* is not an integer, which is a contradiction, so no solution!

Case 3: When $a=9,\ b=7,\ 4a+2b=50$ but $3c=105-63-35=7$, c is not an integer again, no solution!

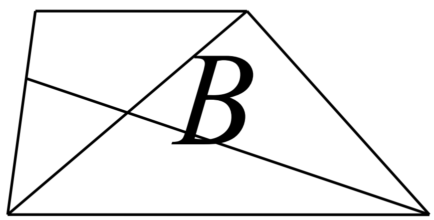
Case 4: When $a=8,\ b=9,\ 4a+2b=50$ but $3c=105-56-45=4$, c can't be an integer, no solution!

Case 5: When $a=8,\ b=8,\ 4a+2b=48$ but $3c=105-56-40=9$, hence $c=3$.

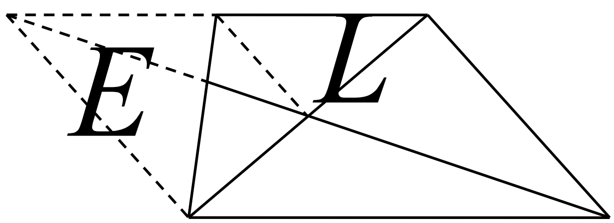
Thus, we conclude the maximum value of $4a+2b$ is 48 and $a+b+c=19$.

Answer: 19

7. In trapezoid *ABCD* of area 630 cm^2, the parallel sides *BC* and *AD* are in the ratio 1:2. *K* is the midpoint of the diagonal *AC* and *L* is the point of intersection of the line *DK* and the side *AB*. What is the area, in cm^2, of the quadrilateral *BCKL*?



【 Solution 1】



B

Let *E* be a point on the extension line of *CB* so that quadrilateral *ADCE* becomes a parallelogram as shown. Since the two diagonal lines of the parallelogram are equally divided, and *K* is the midpoint of *AC*, then *K* must also be the midpoint of *DE* and it follows *D*, *K*, *L* and *E* are collinear. Because $EC=AD=2BC$, then point *B* is the midpoint of the *EC*, this implies *L* is the centroid of ΔACE. Let [*] denote the area of polygon. Since $AD=2BC$, hence $[\triangle ACD]=2\times[\triangle ABC]$, that is

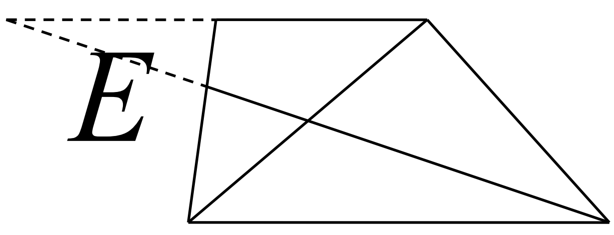
$$[\triangle ABE]=[\triangle ACD]=\frac{1}{3}\times 630\ \text{cm}^2=210\ \text{cm}^2\ ;$$

Because point L is the centroid of ΔACE, so

$$[\triangle ALK]=\frac{1}{6}[\triangle ACE]=\frac{1}{6}\times 420\ \text{cm}^2=70\ \text{cm}^2,$$

so $[BCKL]=210-70=140\,\text{cm}^2$.

【Solution 2】



B

Extend DL and CB so that their intersection will be at point E. Since $BC//AD$, then $\angle ECK = \angle DAK$. We also know that $AK = CK$ and $\angle EKC = \angle DKA$, so that $\Delta ECK \cong \Delta DAK$. Therefore, $EC = AD = 2BC$, it follows $BC = EB$. Similarly, $BC//AD$, then $\angle EBL = \angle DAL$. ; Also from $\angle EKC = \angle DKA$, then $\Delta ELB \sim \Delta DLA$. Hence $\dfrac{EB}{AD} = \dfrac{LB}{AL} = \dfrac{1}{2}$ and this implies $AL = 2LB$. Let [*] denote the area of polygon *. Apply Common Angle Theorem, we obtain

$$\frac{[AKL]}{[ABC]} = \frac{AL \times AK}{AB \times AC} = \frac{\frac{2}{3}AB \times \frac{1}{2}AC}{AB \times AC} = \frac{1}{3}.$$

Because $BC : AD = 1 : 2$, then $\dfrac{[ACD]}{[ABC]} = \dfrac{2}{1}$, so that $[ABC] = \dfrac{630}{3} = 210\,\text{cm}^2$.

Hence, $[BCKL] = [ABC] - [ALK] = \left(1 - \dfrac{1}{3}\right) \times [ABC] = 140\,\text{cm}^2$.

Answer: 140 cm^2

8. Find the number of ways to choose an ordered pair (a, b) of distinct positive integers from 1, 2, 3, … , 100 such that $a < b$ and the product of a and b is divisible by 5.

【Solution 1】

The possible values of a and b can be discussed as follows: If a is a multiple of 5, then the possible values of a are 5, 10, 15, ⋯, 95 which has a total of 19 of them and for each given value of a, b has a total of $100 - a$ possible values:

a	5	10	15	20	25	30	35
b	6 ~ 100	11 ~ 100	16 ~ 100	21 ~ 100	26 ~ 100	31 ~ 100	36 ~ 100
Number of Ordered Pairs (a, b)	95	90	85	80	75	70	65

9.

a	40	45	50	55	60	65	70
b	41 ~ 100	46 ~ 100	51 ~ 100	56 ~ 100	61 ~ 100	66 ~ 100	71 ~ 100
Number of Ordered Pairs (a, b)	60	55	50	45	40	35	30

a	75	80	85	90	95

b	$76\sim100$	$81\sim100$	$86\sim100$	$91\sim100$	$96\sim100$
Number of Ordered Pairs (a, b)	25	20	15	10	5

Therefore, there are $95+90+85+\cdots+5=\frac{19(95+5)}{2}=950$ different ordered pairs (a,b).

If a is not a multiple of 5, then the value of b needs to be greater than a and is a multiple of 5; that is, if $a=5q+r$, $r\le4$, the value of b may be $5(q+1),\ 5(q+2),\ 5(q+3),\ \cdots,\ 100$, so a total of $20-q$ possible values:

a	$1\sim4$	$6\sim9$	$11\sim14$	$16\sim19$	$21\sim24$
b	5, 10, 15,..., 100	10, 15, 20,..., 100	15, 20, 25,..., 100	20, 25, 30,..., 100	25, 30, 35,..., 100
Number of Ordered Pair (a, b)	4×20	4×19	4×18	4×17	4×16

10.

a	$26\sim29$	$31\sim34$	$36\sim39$	$41\sim44$	$46\sim49$
b	30, 35, 40, …, 100	35, 40, 45, …, 100	40, 45, 50, …, 100	45, 50, 55, …, 100	50, 55, 60, …, 100
Number of Ordered Pair (a, b)	4×15	4×14	4×13	4×12	4×11

11.

a	$51\sim54$	$56\sim59$	$61\sim64$	$66\sim69$	$71\sim74$
b	55, 60, 65, …, 100	60, 65, 70, …, 100	65, 70, 75, …, 100	70, 75, 80, …, 100	75, 80, 85, 90, 95, 100
Number of Ordered Pair (a, b)	4×10	4×9	4×8	4×7	4×6

a	$76\sim79$	$81\sim84$	$86\sim89$	$91\sim94$	$96\sim99$
b	80, 85, 90, 95, 100	85, 90, 95, 100	90, 95, 100	95, 100	100

Number of Ordered Pair (a, b)	4×5	4×4	4×3	4×2	4×1

12.

Therefore, there are $4(20+19+18+\cdots+1)=4\times\frac{20(19+1)}{2}=840$ different ordered pairs (a, b).

In summary, there are a total of $950+840=1790$ different ordered pairs (a, b).

【Solution 2】

The possible values of a and b can be discussed as follows:

When a is a multiple of 5, from the Solution 1 we know there are 950 different ordered pairs (a, b).

When b is a multiple of 5, then the possible values of b are $5, 10, \cdots, 100$; so a total of 20 possible values of b. For each given value of b, a has a total of $b-1$ values, so there are number of $4+9+14+\cdots+99=\frac{20(99+4)}{2}=1030$ different ordered pairs (a, b). If a and b are both multiples of 5, they can be listed as follow:

B	100	95	90	85
A	5, 10, 15, ..., 95	5, 10, 15, ..., 90	5, 10, 15, ..., 85	5, 10, 15, ..., 80
Number of Ordered Pair (a, b)	19	18	17	16

13.

B	80	75	70	65	60
A	5, 10, 15, …, 75	5, 10, 15, …, 70	5, 10, 15, …, 65	5, 10, 15, …, 60	5, 10, 15, …, 55
Number of Ordered Pair (a, b)	15	14	13	12	11

14.

b	55	50	45	40	35
a	5, 10, 15, …, 50	5, 10, 15, …, 45	5, 10, 15, …, 40	5, 10, 15, …, 35	5, 10, 15, …, 30
Number of Ordered Pair (a, b)	10	9	8	7	6

b	30	25	20	15	10

a	5, 10, 15, 20, 25	5, 10, 15, 20	5, 10, 15	5, 10	5
Number of Ordered Pair (a, b)	5	4	3	2	1

Therefore, there are $19+18+17+\cdots+1=\frac{19(19+1)}{2}=190$ different ordered pairs (a, b).

In summary, there are a total of $950+1030-190=1790$ different ordered pairs (a, b).

【Solution 3】

We know there are a total of $C_2^{100}=\frac{100\times 99}{2}=4950$ ways of select any 2 numbers from 1 to 100. Since there is a total of 80 numbers from 1 to 100 that are not a multiple of 5, and the numbers of ways to select 2 numbers from these 80 numbers is $C_2^{80}=\frac{80\times 79}{2}=3160$. Therefore, a total of $4950-3160=1790$ different pairs of (a, b) can be found.

Answer: 1790 ways

9. How many increasing arithmetic sequences of the form a, $a+d$, $a+2d$, $a+3d$, … , $a+100d$, where a and d are positive integers, are there such that the last term, $a+100d$, does not exceed 2018? (e.g. 2, 5, 8, 11, … , 302 is one such sequence)

【Solution 1】

From the given information, $a+100d\le 2018$, a and d are positive integers, so that $d\le 20$. Then, for the value of d, the possible values of a can be listed as follows:

d	20	19	18	17	16	...	1
a	1 ~ 18	1 ~ 118	1 ~ 218	1 ~ 318	1 ~ 418	...	1 ~ 1918
Number of Terms	18	118	218	318	418	...	1918

Thus, there is a total of $18+118+218+\cdots+1918=\frac{20\times(18+1918)}{2}=19360$ eligible increasing arithmetic sequence.

【Solution 2】

From the given information, $a+100d\le 2018$, a and d are positive integers, so we know that $a\le 1918$. Then for each possible value of a, the possible values of d can be listed as follows:

a	$1\sim18$	$19\sim118$	$119\sim218$	$219\sim318$
d	$1\sim20$	$1\sim19$	$1\sim18$	$1\sim17$
Number of Terms	18×20	100×19	100×18	100×17

10.

a	$319\sim418$	$419\sim518$	$519\sim618$	$619\sim718$
d	$1\sim16$	$1\sim15$	$1\sim14$	$1\sim13$
Number of Terms	100×16	100×15	100×14	100×13

11.

a	$719\sim818$	$819\sim918$	$919\sim1018$	$1019\sim1118$
d	$1\sim12$	$1\sim11$	$1\sim10$	$1\sim9$
Number of Terms	100×12	100×11	100×10	100×9

12.

a	$1119\sim1218$	$1219\sim1318$	$1319\sim1418$	$1419\sim1518$
d	$1\sim8$	$1\sim7$	$1\sim6$	$1\sim5$
Number of Terms	100×8	100×7	100×6	100×5

13.

a	$1519\sim1618$	$1619\sim1718$	$1719\sim1818$	$1819\sim1918$
d	$1\sim4$	$1\sim3$	$1\sim2$	1
Number of Terms	100×4	100×3	100×2	100×1

Therefore, there are $18\times20+100\times(19+18+17+\cdots+1)=360+19000=19360$ eligible increasing arithmetic sequences.

Answer: 19360

10. An electronic board is divided into 13×13 square lights, all of which are initially switched off. In the first second, the central light and the four corner lights are switched on. In each subsequent second, the squares that have a common side with an already lit square are simultaneously switched on. This process is continued until all 169 lights are switched on. In the following second, all lights are switched off. Now, the above operation is repeated. How many square lights will be switched on at the 60th second? (For example: At the 2nd second, 12 square lights will be switched on.)

【Solution】

1	2	3	4	5	6	7	6	5	4	3	2	***1***
2	3	4	5	6	7	6	7	6	5	4	3	2
3	4	5	6	7	6	5	6	7	6	5	4	3
4	5	6	7	6	5	4	5	6	7	6	5	4
5	6	7	6	5	4	3	4	5	6	7	6	5

6	7	6	5	4	3	2	3	4	5	6	7	6
7	6	5	4	3	2	***1***	2	3	4	5	6	7
6	7	6	5	4	3	2	3	4	5	6	7	6
5	6	7	6	5	4	3	4	5	6	7	6	5
4	5	6	7	6	5	4	5	6	7	6	5	4
3	4	5	6	7	6	5	6	7	6	5	4	3
2	3	4	5	6	7	6	7	6	5	4	3	2
1	2	3	4	5	6	7	6	5	4	3	2	***1***

In the above figure, the number in each small square represents the changes in every second of the bulb (from dark to bright) in that position during the first seven seconds. The figure shows:
In the 1st second, a total of $4+1=5$ lights turned from dark to bright.
In the next second, a total of $4\times 2+4\times 1=12$ lights turned from dark to bright.
In the 3rd second, a total of $4\times 3+4\times 2=20$ lights turned from dark to bright.
In the 4th second, a total of $4\times 4+4\times 3=28$ lights turned from dark to bright.
In the 5th second, a total of $4\times 5+4\times 4=36$ lights turned from dark to bright.
In the 6th second, a total of $4\times 6+4\times 5=44$ lights turned from dark to bright.
In the 7th second, a total of $4\times 7=28$ lights turned from dark to bright.

At this time, all 169 bulbs on the electronic panel are lighted, so in the 8th second, all the lamps are changed from bright to dark. At this point, it can be judged that the situation of the 9th second is the same as that of the 1st second; that is, at every second, the number of lights that are changed from dark to bright is 5, 12, 20, 28, 36, 44, 28, 0. The eight numbers are continuous repeated in the same order. Since $60=8\times 7+4$, the situation at the 60th second is the same as the 4th second; that is, a total of 28 bulbs change from dark to bright.

Answer: 28 bulbs