Game of Four Numbers (continued)

That proves that every 4-number game of whole numbers must eventually stop. But there are still many questions to investigate. Here are a few for you to think about:

1. How many steps are needed? Is there some row that needs 20 steps to get to zeros? Is there one that needs 100 steps?

What's the longest number of steps needed for a row whose entries are all < 100 ?

2. What about other row sizes? For instance, what happens with rows of three numbers? Or with rows of five or six? What sorts of patterns occur as the game continues?

3. What happens when more general numbers are used in the four-number game? For instance, work out the game for $(0, 1, 6, \pi)$. Surprisingly this one goes to zeros in only four steps. Investigate some other non-integer examples. What happens?

For those cases our proof that the game ends in zeros no longer works. (Dang!) Are there any cases when the four-number game is infinite?

Note: Dozens of technical papers have been written about the four-number game. Those patterns are also called "Ducci sequences" after the mathematician who invented this game in the 1930s.

Here are a few references:
M. Lotan, A problem in difference sets, Amer. Math. Monthly 56 (1949) 535-541.
(Proves the general result for 4 real numbers.)
E. Berlekamp, The design of slowly shrinking squares, Math. Comp. 29 (1975), 25-27.
(Same result with fewer details.)
L. Meyers, Ducci's four-number problem: a short bibliography, Crux Math. 8 (1982), 262-266.
W. Webb, The length of the four-number game, Fibonacci Quart. 20 (1982) 33-35.
(The row $(t_n, t_{n-1}, t_{n-2}, t_{n-3})$ has a game of length about $3n/2$, and no row involving numbers of that size can last longer. Here t_n is a Tribonacci number!)
R. Brown and J. Merzel, The length of Ducci's four-number game, (preprint).

Daniel Shapiro is a professor of Mathematics at Ohio State University. He is also the director of the Ross Mathematics program and a frequent contributor to Math Explorer.

Dear *Math Explorers*,

We end our year of math exploration with our main article, "Game of Four Numbers." Our author, Daniel Shapiro, shares an interesting sequence, named after the mathematician, Ducci who invented the four-number game. We hope the summer will give you extra time to work on the intriguing questions our author poses in Math Odyssey. The fascinating life of Poincaré is highlighted in our biography. Learn how an outstanding problem posed by Poincaré is one of seven problems that has a valuable reward, if solved!

We at *Math Explorers* wish you a relaxing summer and look forward to having you join in more math explorations in the fall!

Sincerely,
Hiroko K. Warshauer
Hiroko K. Warshauer, Executive Editor

Vol. 7.4 Summer 2005

Math Explorer

MATHEMATICS AND GAMES

POINCARÉ CONJECTURE HAS VALUE$

Game of Four Numbers

WHAT'S THE DIFFERENCE?

Math Explorer

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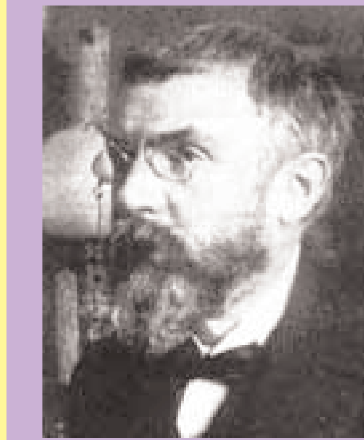

Jules Henri Poincaré

by Christopher Johnson

Jules Henri Poincaré is one of the great geniuses of all time. As such he fit the geek stereotype perfectly. He wore clunky glasses and lacked coordination. To add to his awkward physical nature, he suffered illness for a time from a bacterial toxin that affects his heart and nervous system. His outward appearance contrasted the many beautiful ideas he originated during his short life. It is his ideas for which he is remembered.

His family helped build a solid educational foundation. His French parents were educated and talented. After his birth in 1854, his mother nurtured him through early education. In high school he excelled in mathematics, winning first prize in a math competition between top students from all the high schools in France.

He continued to improve his understanding of math in college. Some famous mathematicians were teachers and friends of Poincaré, including Hermite, Liousville, and Lorentz. At this time, Europe was bustling with new ideas in mathematics. This mathematical environment played a large role in helping to establish his lasting fame.

Poincaré believed it was more important to invent mathematical ideas than to prove them with logic. He said, "Logic remains barren unless fertilized by intuition." He solved many difficult problems, but perhaps his greatest fame stems from a question. Today, we call this question the Poincaré Conjecture, which means we think we know the answer, but we're not sure. There is a one million dollar award from the Clay Mathematics Institute to anyone who can answer this 101-year old question.

As Poincaré grew more successful, he became interested in the way his mind worked. He claimed that when he couldn't figure something out right away, he would stop thinking about it for a while. Then, suddenly, while preoccupied with something else, he would have a "flash of cognition" and would know the answer. It is said that he usually started his answers from basic principles. This approach allowed him to work with experts in several branches of mathematics.

Poincaré is the last mathematician who could solve difficult problems in many branches of mathematics. This puts him in an elite class with mathematicians like Gauss and Euler. The important dignitaries that attended his funeral in July of 1912, including the President of France, stands as a final evidence of the legacy he created even during his lifetime. Poincaré died at the age of 58.

Christopher Johnson teaches mathematics at Texas State University. He enjoys unravelling the exciting mysteries that abound in the world of mathematics

Bulletin Board

Math Olympiad

Want to have a Math Olympiad team in your school? Learn more by visiting http://www.moems.org/program.htm

A Million Dollar Math Question

The Poincaré Conjecture is one of the seven greatest unsolved mathematical puzzles of our time. Called the Millennium Prize Problems, the Clay Mathematics Institute will award whomever solves (once verified by experts) one of the seven problems $1 million in prize money.
The Poincaré Conjecture arises in the field of topology, sometimes referred to as "rubber-sheet geometry."
Visit the Clay Institute website for the statement of the conjecture.
http://www.claymath.org/millennium/Poincare_Conjecture/

Wonder if the solution will be found in our lifetime? Maybe you can one day contribute to the solution. Keep studying mathematics. Learning mathematics can be quite rewarding!

Math Challenges for Families!

A CD-ROM from the National Council of Teachers of Mathematics entitled: "**Figure This! Math Challenges for Families: Take a Challenge**!" provides real-world mathematics problems students and their families can solve. Available in both English and Spanish versions. For information call 800-235-7566 or visit www.nctm.org

Chuckle Bytes

What do you call one millionth of a mouthwash?

A microscope

What do you call a basic unit of laryngitis?

A hoarsepower

Correction

The article, "Angles, Triangles, and Ears," in Winter, 2004 contained an error .
Please visit our website, www.txstate.edu/mathworks to see the correction.

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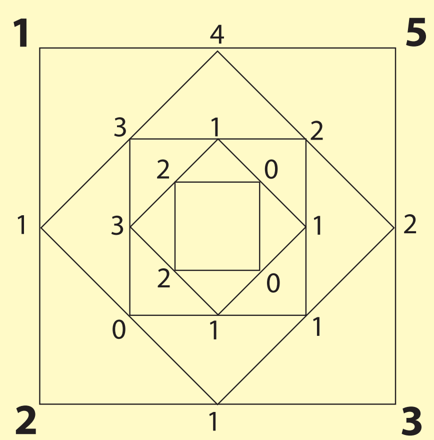
FOUR NUMBERS

by Daniel Shapiro

Choose four numbers and place them at the corners of a square. At the midpoint of each edge, write the difference of the two adjacent numbers, subtracting the smaller one from the larger. This produces a new list of four numbers, written on a smaller square.



What happens when this process is repeated?

Here are a few steps, starting with the four numbers 1, 5, 3, 2 around the largest square, and proceeding inwards. Once you see how it works, it's easier to display the game more compactly as a table:

1	5	3	2
4	2	1	1
2	1	0	3
1	1	3	1
0	2	2	0
2	0	2	0
2	2	2	2
0	0	0	0

After seven steps the numbers become all zeros. Let's try two more examples.

1	3	8	17
2	5	9	16
3	4	7	14
1	3	7	11
2	4	4	10
2	0	6	8
2	6	2	6
4	4	4	4
0	0	0	0

1	2	2	5
1	0	3	4
1	5	1	3
2	2	2	2
0	0	0	0

Each example ends with a row of zeros after a few steps. Take a couple of minutes to try out a few 4-number patterns for yourself. . .

Are there any examples that don't end with a row of zeros?

One way to investigate this question is to generate lots of examples. We could ask everyone we know to work out fifty examples. Or you could write a computer program to compute examples. But lists of examples can never prove that the process will *always* end in a row of zeros.

To find a proof, we start by defining terms more carefully. If Q = *(a, b, c, d)*, the derived row Q′ is obtained by taking differences, ignoring minus signs. The first entry of Q′ will be either *a - b* or *b - a*, whichever one is not negative. That entry is the absolute value $|a - b|$. With this terminology, if Q = (*a, b, c, d*), the derived row is:

$$Q' = (|a-b|, |b-c|, |c-d|, |d-a|)$$

If we analyze the general situation directly, the cases and sub-cases proliferate: Which of the numbers is largest? Which of the differences is smallest? We use a more indirect approach.

OBSERVATION. If Q ≠ (0, 0, 0, 0), then Q′ seems to be smaller than Q.

If this is always true, we can prove that our game must end in a row of zeros. Suppose Q = *(a, b, c, d)* is given with non-negative integer entries. Repeat the process several times, obtaining rows Q′, Q″, Q‴, Q⁗, . . . By the Observation the entries in those derived rows get smaller and smaller. Eventually they become zero, since a decreasing sequence of non-negative integers cannot go on forever.

But is that Observation true? In the examples the numbers get smaller as the game is played, but what exactly does it mean for one row to be "smaller" than another? For instance, is (1, 0, 4, 12) smaller than (3, 3, 5, 4)? Here's another example:

If Q = *(4, 0, 0, 0)* then Q′ = *(4, 0, 0, 4)*. Here the size did NOT decrease (whatever measure of size we use).

To clarify the "size" of a row, let's consider the maximal entry:

If Q = *(a, b, c, d)*, let **max(Q)** be the largest of the four numbers in Q.

Since our process uses subtraction, the largest number in Q cannot increase. In mathematical terms, this says: If Q′ is derived from Q then: max(Q′) ≤ max(Q).

Those maximal values might be equal (as seen when Q = *(4, 0, 0, 0)*). That can happen only when there is a zero in the row Q. Since equality of maximal entries can happen we have to work a little more, running the game a few steps.

CLAIM: For any row R, at least one of the rows R or R′ or R″ or R‴ or R⁗ has all entries even.

This Claim is proved by considering a few cases. Write "e" for an even number and "o" for an odd number, to track different cases. For instance the row Q = (4, 2, 1, 1) becomes *(e, e, o, o)* and we find the derived row must be: Q′ ≈ *(e, o, e, o)*. (Why?) Similarly, Q″ ≈ *(o, o, o, o)*, and Q‴ ≈ *(e, e, e, e)*, which is all even. There are several more *e* & *o* cases to work out, but we leave them for you to investigate.

Now we can prove that the game must eventually stop at (0, 0, 0, 0) for any initial row Q, no matter how large the entries of Q are. To start, run the game for a few steps until reaching some derived row S which is all even (using the Claim). That row can be written as S = *(2w, 2x, 2y, 2z)* for some whole numbers *w, x, y, z*. Let T = (*w, x, y, z*) which is just 1/2 S. The steps of the game applied to S exactly match the steps applied to T. (Why?)

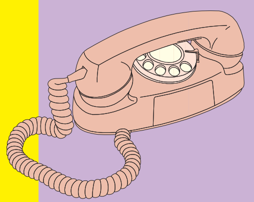
Then to analyze the game we can replace S by T. Note that max(T) is definitely smaller than max(S), at least if S was non-zero. Then the previous idea works: Continue the game, but each time we reach an all even row, factor out another 2. If a row of zeros never appears, the maximal entries of the factored rows provide a decreasing list of positive integers that never ends. That's impossible!

(continued on back cover)

Puzzle Page

Math Explorers: We want to print your work! Send your original math games, puzzles, problems, and activities to:
Texas Mathworks, 601 University Dr., San Marcos, TX 78666

There are six gossips who like to share information. Whenever one of them calls another, by the end of the conversation they both know everything that the other one knew beforehand. One day, each of the six finds out a juicy piece of gossip. What is the minimum number of phone calls needed before all six of them know all six of these juicy tidbits?



Fill in the missing squares so that the rows, columns and diagonals all add up to the same number.

32	19		8
10	25		
9			
35	16		11

Word Search

Forwards or backwards, up, slanted, or down.
Where can the words in this puzzle be found?

- Poincare
- Conjecture
- Sequence
- Difference
- Maximal
- Decrease
- Process
- Zero
- Negative

Y	A	Q	P	A	W	D	I	E	M	S	Z	E	Q
B	S	X	J	T	H	R	S	V	E	T	L	C	N
K	F	W	F	W	E	A	S	V	A	N	R	N	P
F	D	K	M	Z	E	R	M	P	D	J	C	E	R
B	N	A	J	R	M	Q	A	A	V	V	C	U	O
X	J	X	C	B	X	H	O	C	X	O	D	Q	C
B	N	E	G	A	T	I	V	E	N	I	C	E	E
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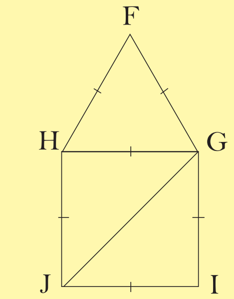
M	M	F	I	U	R	O	I	Y	Z	F	V	P	F
M	L	Q	R	E	C	N	E	R	E	F	F	I	D
Q	U	E	D	C	V	Q	I	N	R	X	M	K	J
C	K	G	J	N	D	H	F	A	F	P	K	H	C

The following problems appeared in the Australian Mathematics Competition for the WestPac Award July, 2004.



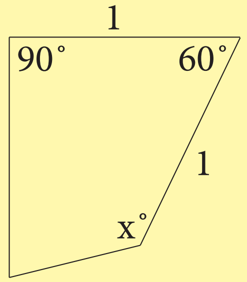
1. An equilateral triangle FGH sits on top of a square GIJH as shown. Find the measure of angle FGJ.

2. Josephine and Jeremy run a 200-meter race which Josephine wins by 10 meters. Jeremy suggests that they run another race, with Josephine starting 10 meters behind the starting line. Assuming they run at the same speeds as in the first race, who will win this second race and by what distance?

3. The integers 1, 2, 3, …, 100 are written on the board. What is the smallest number of these integers that can be wiped off so that the product of the remaining integers ends in 2?

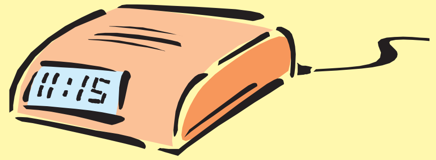
4. Two six-sided dice are tossed and the difference between the numbers appearing uppermost on the dice is recorded. What difference is most likely to occur?

5. Consider the quadrilateral and find the value of x given the information provided.



6. A six-digit number is represented by 1vwxyz, where 1(one), v, w, x, y, z are its digits. If this number is multiplied by 3, the result is vwxyz1. Find the original six-digit number.

7. A digital clock displays hours as a 2-digit number and minutes as 2-digit number. What is the total time in minutes that the digit 2 was visible on the face of the watch from 10:00 to 11:30 one morning?



8. Seven numbers, each 1 or -1, are listed in a row in such a way that adding the numbers successively from left to right never gives a negative answer. For example,
1, -1, 1, 1, -1, -1, 1 has successive sums 1, 0, 1, 2, 1, 0, 1 and is valid, while
1, 1, -1, -1, -1, 1, 1 has successive sums 1, 2, 1, 0, -1, 1, and is not valid.
How many valid lists are there?

Answer: **1)** 105° **2)** Josephine by 0.5 meter **3)** 21 **4)** 1 **5)** 135° **6)** 142857 **7)** 27 **8)** 35