MATH ODYSSEY by Christopher Johnson

Uncountable Sets

If we place a point between 0 and 1 on the number line, it represents a real number greater than 0 and less than 1. One way to represent this number is as a decimal. We can begin by dividing the interval from 0 to 1 into 10 equal size pieces.

Next we get out the 10x microscope and divide the interval from 0.6 to 0.7 into 10 equal size pieces. We zoom in again and divide the interval from 0.65 to 0.66 into 10 equal size pieces. If we continue this process forever, the number represented by our dot in the picture will be expressed as 0.659... in an *infinite decimal expansion.* We can carry out this process for every real number greater than 0 and less than 1.

We create a set called the set of real numbers greater than 0 and less than 1, and we write this set in symbols as (0,1). Let's assume (0,1) is countable, then there is a one-to-one correspondence between (0,1) and the set of natural numbers. With this assumption, we have the following correspondence, which we consider as a list of the set (0,1).

Natural Numbers		Number in (0,1) expressed in an infine decimal expansion
1	⟷	0.52193583248...
2	⟷	0.78326941585...
3	⟷	0.12358974656...
4	⟷	0.00125658456...
5	⟷	0.98548765255...
...	⟷	...

Next we construct a set of real numbers that is between 0 and 1 and not on this list! Some rational numbers have two distinct infinite decimal expansions. For example, 1 can be represented as 1.00000.....and 0.99999.... To avoid constructing a number with two different representations, we will only use digits from 1 to 8.

Let's begin constructing our number by looking at the first decimal position's digit on our number from our list. We see 5, so we use 6 in our number's first decimal digit. The second decimal position number needs to be different from 8, so we choose 7. For the third decimal position we choose 2; the fourth, 3; the fifth, 7. We continue in this way, choosing our nth decimal position number different from the nth decimal position of the number in (0,1) that corresponds to the natural number n. We have constructed the number x = 0.67237...

Notice that x is different from each number on our list for two reasons. First we chose it to differ in at least one decimal place from each number on the list. Second, we chose it to have a unique decimal expansion as we avoided an infinite string of zeros or nines. We assumed our list contained all numbers in (0,1). The number x is in (0,1). The number x must be on the list. At the same time, we have shown x is not on the list. We are at a logical crossroads. We must have been wrong in our assumption that the set (0,1) was countable. The only logical conclusion, then, is that (0,1) is uncountable!

Dear Math Explorers,

Welcome to another exciting year of Math Explorer! We hope the variety of problems and puzzles, as well as the article and biography provide challenges and opportunities for mathematical growth. This issue's topic of infinity will be just the beginning for your continued exploration in this fascinating area of study.

Please take time to return the enclosed survey. We will send our first 50 respondents a special gift!

Sincerely,

Hiroko K. Warshauer

Hiroko K. Warshauer, Executive Editor

Vol. 7.1 Fall 2004

Math Explorer

MATHEMATICS AND INFINITY

INFINITY **SUPER**-SIZED

Cantor Sets Intrigue

One-to-One Correspondence

Math Explorer

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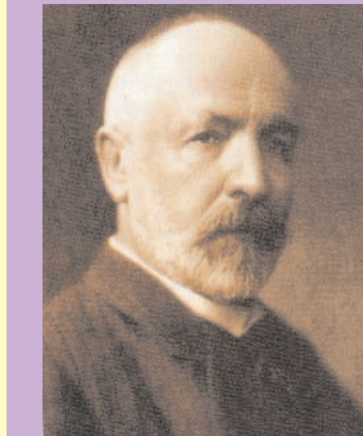

Georg Ferdinand Ludwig Philipp Cantor

by Hiroko Warshauer

Georg Cantor is best known for his works in set theory and distinguishing the sizes of infinity. There is even a set , called the Cantor Set, that is named after him.

Though Cantor is called a German mathematician, he was actually born in St. Petersburg, Russia on March 3, 1845. In 1856, young Cantor, then 11, his Danish merchant father, George Waldemar Cantor, and his Russian musician mother, Maria Anna Bohm, moved to Germany. Cantor continued his studies in Germany and in 1867 earned a doctorate in number theory from the University of Berlin.

In 1873 and 1874, Cantor showed there are different sizes of infinity. He proved that the set of rational numbers has a one-to-one correspondence with the set of natural numbers and thus can be called **countable.** The set of real numbers, however, is **uncountable** and of a larger size. Terms such as countable and uncountable, denumberable and non-denumerable, or $\aleph_0$ (aleph naught) and $\aleph_1$ (aleph 1) are used to distinguish the sizes of infinity.

Along with his paper on countable and uncountable sets, Cantor published papers on the uniqueness of representation of a function as a trigonometric series, introduction to set theory and several paradoxes in set theory. Cantor worked with many famous mathematicians of his time, including Kronecker, Dedekind, Mittag-Leffler and Weierstrass. However, some of these relationships grew strained and created difficulties for Cantor in gaining mathematical acceptance of his ideas. Cantor also suffered from depression, which affected his ability to work. During those bouts, he would turn his attention away from mathematics to philosophy and Shakespeare.

On August 9, 1874 Cantor married Vally Guttmann. Their family grew to include six children, though their youngest son died tragically at the age of 13. Personal tragedy and the resistance of the mathematical community to his ideas contributed to the deterioration of his mental condition. Cantor retired in 1913, just prior to World War I, when conditions in Germany were difficult. He entered a sanatorium and died of a heart attack on January 6, 1917.

References: http://en.wikipedia.org/wiki/Georg_Cantor
http://www-gap.dcs.st-nd.ac.uk/~history/Mathematicians/Cantor.html
http://www.factindex.com/g/ge/georg_cantor.html
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Hiroko Warshauer teaches mathematics at Texas State University-San Marcos and is also the Executive Editor of Math Explorer.

Bulletin Board

Mathworks Team Competes in Hong Kong

Jeffrey Chen, Kaitlyn McClymont, Elizabeth Tsai and Michael Zhang brought back the Po Leung Kuk Cup, awarded to the top non-Asian team at the 8th Primary Math World Contest held in Hong Kong this July. This is the third Texas Mathworks team to win this award! Team leaders Sam Baethge and Hiroko Warshauer escorted the team.

Campers Enjoy Junior Summer Math Camp

Over 200 students participated in the 10th Mathworks Junior Summer Math Camp held June 7-18 at Hernandez Intermediate School in San Marcos, TX. Students from 3rd grade through 8th grade took classes that introduced them to algebra, geometry and problem solving. For information on next year's camp, contact Mathworks at *mathworks@txstate.edu*

Math Bytes-Check it out!

www.archimedes-lab.org offers fun puzzles, optical illusions and instructions for making your own puzzles. You can practice your Italian and French, too!

Mathforum.org/Isaac/mathhisto.html provides weekly problems, biographies and much more.

MATHCOUNTS is a national math enrichment, coaching and competition program that promotes middle school mathematics. For more information, contact *info@mathcounts.org*

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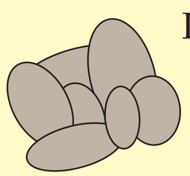
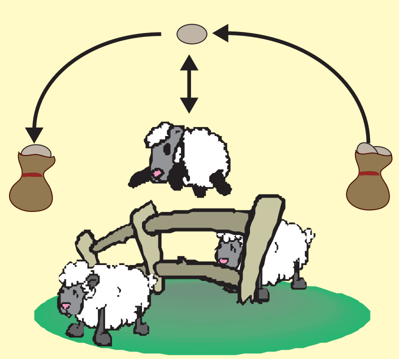
Infinity

by Christopher Johnson

Before humans could count, shepherds needed to keep an eye on how many sheep were in their flock. They would carry around a sack containing one pebble for each of their sheep. At dusk, for each sheep entering their fold they would take a pebble out of their sack and put it in an empty sack. If any pebbles were left in the first sack, they knew they had lost one or more sheep that day. This is similar to what modern mathematicians call **one-to-one correspondence**.

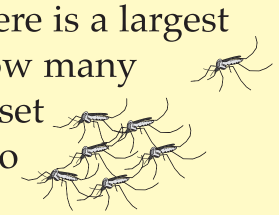
In place of the pebbles, humans made up names that corresponded to certain numbers of things. We even created an entire naming system so that any number of things would correspond to a unique name. With this naming system, we can now count as high as we want. We can compare the number of sheep in a shepherd's flock in the morning to the number of sheep in his flock at dusk. We can recognize that these sets are either the same or that one is larger than the other.





Since a set is just a collection of things, we can collect all the numbers we use to count with together into what we call the set of **natural numbers**. We can even collect all the natural numbers, zero, and all the negative natural numbers together into the set of **integers**. Does the set of integers have more numbers in it than the set of natural numbers?

If we think really hard, we can imagine a set containing all the mosquitoes in the Amazon. Though really big, this set has a finite size. This means there is a largest number that describes how many things are in the set. The set of natural numbers has no such largest number, and so it is considered **infinite,** or not finite.



How can we say one infinite set is larger than another infinite set? For each number, such as 1 in the set of natural numbers, there are two related numbers 1 and –1 in the set of integers. One might say the set of integers is at least twice as large as the set of natural numbers. Two times three equal to six makes sense, but does two times infinity make sense? What does two times infinity equal?

When complicated things don't make sense, we sometimes have to go back to where it all began. In the case of counting, we go back to the one-to-one correspondence of the shepherd. We will call a set of infinite size **countable**, if we can place it in a one-to-one correspondence with the set of natural numbers. Otherwise, we will call such a set **uncountable.** Look how the set of integers can be put in a one-to-one correspondence with the set of natural numbers, just like the sheep were with the pebbles.

Natural Numbers	1	2	3	4	5	6	7	8	9	...
	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
Integers	0	-1	1	-2	2	-3	3	-4	4	...

What natural number matches up with the integer –51 in the one-to-one correspondence above? What integer matches up with the natural number 100? Would you consider the set of integers countable or uncountable?

We can collect all the ratios of one integer to another integer (except zero) into a set called the set of rational numbers. Since we can compare fractions, let's think of a few fractions greater than –1 and at the same time less than 1: –1/2, –1/8, 0, 1/2, 3/4, 7/8, 15/16, and many more. The only integer between –1 and 1 is 0. It seems like the set of rational numbers must be much larger than the set of integers. Ask your favorite math teacher to show you how in fact the set of rational numbers is also countable!

Just like all of our sets so far, perhaps all sets can be put in a one-to-one correspondence with the set of natural numbers. As it turns out, when we study more sets of numbers we will discover the set of irrational numbers. We can show this is uncountable, and that's just the beginning.

Since it is confusing to compare sets of infinite size, the simple one-to-one correspondence of the shepherd made more sense than our advanced system of counting. In mathematics, simplifying complicated things often clears up the confusion. But what is the purpose of comparing sets of infinite size anyway? Let's go keep watch not over sheep but for sets of uncountable size (S–U–S). If we find one (see Math Odyssey on page 8), maybe the use will become clear.

Christopher Johnson is a graduate student in mathematics at Texas State University-San Marcos. He enjoys unravelling the exciting mysteries that abound in the world of mathematics.

Puzzle Page

Math Explorers: We want to print your work! Send your original math games, puzzles, problems, and activities to:
Texas Mathworks, 601 University Dr., San Marcos, TX 78666

A woman and her husband hosted a party for four other couples. The hostess asked everyone to shake hands and introduce themselves to each other. Of course, no one shook hands with their spouse. At some point, the hostess stopped them and asked each person how many hands he or she had shaken. Each person gave a different response. What was the response of her husband?

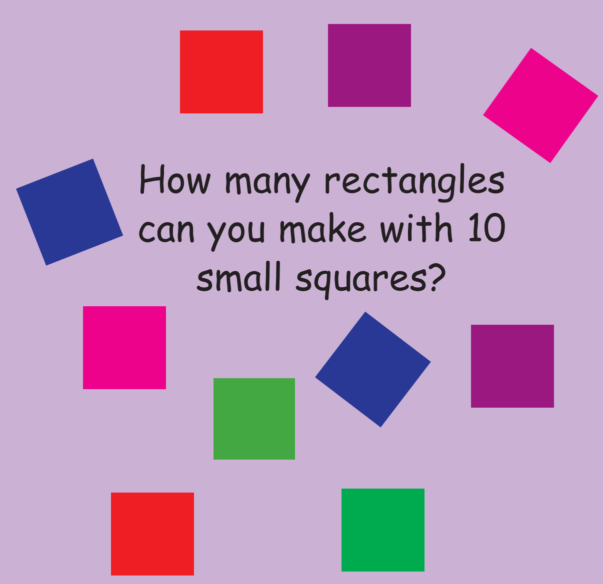
Divide 100 marbles into a number of bags so that I can ask for any number of marbles between 1 and 100, and you can give me the proper amount by giving me a certain number of bags without opening any of them. What is the minimum number of bags you will require?

Word Search

Forwards or backwards, up, slanted, or down.
Where can the words in this puzzle be found?

Cantor
Diagonalization
Irrational
Correspondence
Infinite
Rational
Countable
Integer

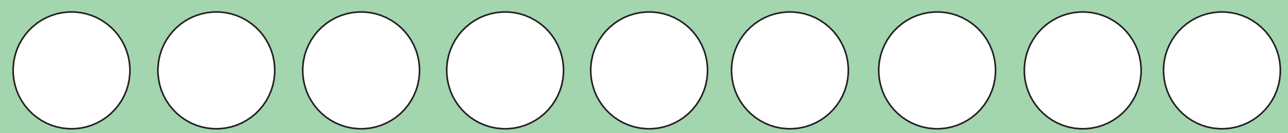
C	C	X	E	O	N	B	J	P	H	X	B	R	A
K	O	M	Y	T	O	Y	K	O	S	X	E	C	B
S	R	Q	U	K	I	T	Q	S	D	G	D	I	T
T	R	Z	A	J	T	N	L	R	E	Y	R	L	O
A	E	R	U	R	A	C	I	T	Q	R	S	A	A
E	S	G	L	Z	Z	S	N	F	A	P	D	N	H
F	P	G	V	I	I	I	V	T	N	F	U	O	G
Y	O	O	T	Q	L	G	I	S	M	I	Z	I	S
V	N	Q	E	I	A	O	K	U	H	Q	E	T	W
J	D	R	C	A	N	T	O	R	F	O	C	A	G
K	E	I	Y	A	O	K	A	K	E	A	S	R	Z
R	N	M	L	D	G	J	U	D	G	R	A	C	L
G	C	E	L	B	A	T	N	U	O	C	F	B	P
C	E	Z	E	E	I	R	K	P	Z	N	K	Y	B
C	K	G	J	N	D	H	F	A	F	P	K	H	C



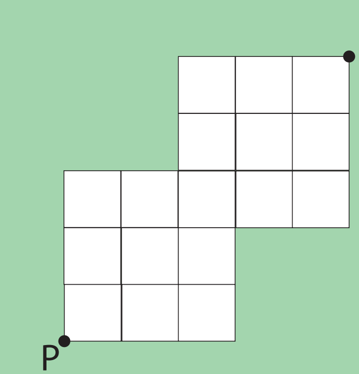
How many rectangles can you make with 10 small squares?

PROBLEMS PAGE

*1. Arrange the digits 1-9, without repetition, in the circles below in such a way that:
All the digits between 1 and 2 add up to 6.
All the digits between 2 and 3 add up to 14.
All the digits between 3 and 4 add up to 38.
All the digits between 4 and 5 add up to 9.
Find the smallest value of this 9-digit number.



2. At a Halloween party consisting of boys, girls, adult women and adult men, there were: 14 girls, 11 adults without costumes, 14 women, 10 girls with costumes, 26 people without costumes, 8 women with costumes and 15 males with costumes. How many people were at the party?



*3. How many shortest paths are there from P to Q?

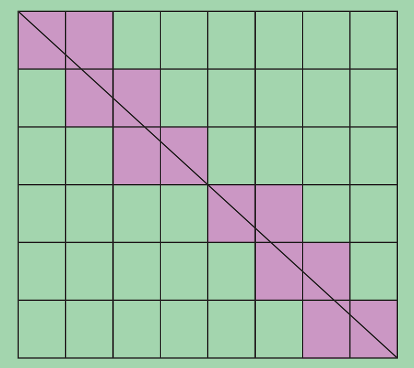
*4. Let y = 1 x 2 x 3 x ... x 20. What is the sum of the last 5 digits of y?

*5. When a certain two-digit number, AB, is added to another two-digit number, BA, the sum is a perfect square. Find the sum of all such possible two-digit numbers.

*6. There are nine fractions between 1/5 and 1/2 where the difference between any two successive fractions is constant. Find the sum of these nine fractions.

7. Jennifer wrote a list of consecutive whole numbers starting with the number 1. She wrote 261 digits. What was the last whole number she wrote?

*8. In a rectangle made up of 2004 x 4002 square units, how many square units can a diagonal line pass through? (For example, the figure below shows a diagonal line of a 6 x 8 rectangle that passes through 12 square units.)



** These problems appeared in the 8th Primary Mathematics World Contest held in Hong Kong July, 2004.*

Answer: 1) 371628594 2) 59 3) 200 4) 4 5) 484 6) 77/20 7) 123 8) 600