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A characterization of oriented hypergraphic balance via signed weak walks

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ABSTRACT

An oriented hypergraph is a hypergraph where each vertex-edge incidence is given a label of $+1$ or -1, and each adjacency is signed the negative of the product of the incidences. An oriented hypergraph is *balanced* if the product of the adjacencies in each circle is positive.

We provide a combinatorial interpretation for entries of kth power of the oriented hypergraphic Laplacian via the number of signed weak walks of length k. Using closed weak walks we prove a new characterization of balance for oriented hypergraphs and matrices that generalizes Harary's Theorem for signed graphs.

1. Introduction

An *oriented hypergraph* is a signed incidence structure where each vertex-edge incidence is given a label of $+1$ or -1, and each adjacency is signed the negative of their incidence product. A *signed graph* is an oriented hypergraph where each edge is con-

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tained in at most two incidences, while a graph can be considered as a signed graph where all adjacencies are positive and each edge is contained in exactly two incidences. An oriented hypergraph is said to be *balanced* if the product of the adjacencies in each circle is positive. In Harary's seminal paper on signed graphs in 1953 he provided the following equivalent condition for balance in signed graphs to model social interactions using signed paths:

Theorem 1.1. *(See Harary [6].) A signed graph is balanced if, and only if, for each pair of vertices v and w all vw-paths have the same sign.*

Harary's Theorem is not true for hypergraphs with edges of size 3 or greater. We relax the path condition and provide a non-trivial oriented hypergraphic generalization of Harary's Theorem.

The concept of a balance, as well as identifying equivalent conditions, is critical to the structure of many combinatorial optimization and programming, for a proper introduction see [4,10]. The (non-oriented) balanced hypergraph was introduced by Berge in 1970 [1] as one of a number of different generalizations of bipartite graphs; this was further generalized to balanced $\{0, \pm 1\}$-matrices by Truemper in 1982 [13], and to incidence oriented hypergraphs by Shi in 1992 [12] and Rusnak in 2013 [9]. The benefit of working with the current oriented hypergraphic model is that it unifies Harary's work on signed graphs with the current theory of balanced matrices, and provides a translation between many graphic, hypergraphic, and balanced matrix theorems.

It was shown by Fulkerson et al. [5] that the condition of balance in $\{0, 1\}$-matrices is equivalent to integrality of set covering, set packing, and set partitioning polytopes as well as total dual integrality of linear systems, these results were extended to $\{0, \pm 1\}$-matrices by Conforti and Cornuéjols in [2]. Berge also provided the following characterization of balance, which was also generalized to balanced $\{0, \pm 1\}$-matrices by Conforti and Cornuéjols in [2]:

Theorem 1.2. *(See Berge [1].) A $\{0, 1\}$-matrix $\mathbf{M}$ is balanced if, and only if, every submatrix of $\mathbf{M}$ is bicolorable.*

These generalizations can be trivially incorporated into oriented hypergraphs by regarding the given matrix as the incidence matrix of an oriented hypergraph. However, a generalization of Harary's Theorem would provide a characterization of balance via new hypergraphic structures and avoid examining matrices. A structural characterization of obstructions to balance was given by Truemper in [14], while Conforti, Cornuéjols, and Rao's famous work [3] on recognizing balanced $\{0, 1\}$-matrices in polynomial time won the 2000 Fulkerson Prize. More recently balanced matrices were generalized to oriented hypergraphs to examine integrated circuits and various applications to VLSI via minimization [11,12]. Additionally, the concept of balance is central to the characterization of the matroid structure of signed graphs [15–17] as well as oriented hypergraphs as introduced in [9].

It was shown in [8] that many results from algebraic graph theory extend to incidence-simple oriented hypergraphs using the oriented hypergraphic incidence, adjacency, degree, and Laplacian matrices. The familiar Laplacian equation

$$L = \mathrm{HH}^T = D - A,$$

as well as the entries of A^k corresponding to signed k-walks, both hold for oriented hypergraphs. More importantly, it was shown that the entries of the Laplacian correspond the newly introduced concept of a signed weak 1-walk — a generalization of a walk that allows loop-like backsteps along the same incidence. Weak walks provide a unified combinatorial object to discuss the entries of any of the oriented hypergraphic matrices. It is important to note that in order to maintain consistency with matroid theoretic results from graph theory and signed graph theory the incidence and adjacency matrices may produce values of 0 for offsetting positive and negative incidences or adjacencies.

We provide three new results in this paper. First, we make a minor improvement to the results appearing in [8] by removing the requirement of incidence-simplicity, showing that the entries of A^k correspond to signed k-walks in any oriented hypergraph. Second, we show that the entries of L^k correspond to signed weak k-walks, demonstrating the combinatorial difference between the adjacency and Laplacian matrices is one of weakness in walks. Finally, we obtain an oriented hypergraphic generalization of Harary's Theorem for signed graphs via self-intersecting weak walks, providing an equivalent characterization of balance for oriented hypergraphs.

2. Background

2.1. Oriented hypergraphs

For a broader introduction to the theory of oriented hypergraphs see [7–9]. An *oriented hypergraph* is a quadruple $(V, E, \mathcal{I}, \sigma)$ where V and E are the sets of vertices and edges, $\mathcal{I}$ is the set of incidences, and $\sigma : \mathcal{I} \to \{+1, -1\}$. The set of incidences is determined by a function $\iota : V \times E \to \mathbb{Z}_{\geq 0}$, and an *incidence* is a triple (v, e, k), where v and e are incident and $k \in [\iota(v, e)]$; the value $\iota(v, e)$ is called the *multiplicity* of the incidence. An oriented hypergraph is *incidence-simple* if all incidence multiplicities are less than or equal to 1.

A *weak walk* is a sequence $\widetilde{W} = a_0, i_1, a_1, i_2, a_2, i_3, a_3, \ldots, a_{n-1}, i_n, a_n$ of vertices, edges and incidences, where $\{a_k\}$ is an alternating sequence of vertices and edges, and i_h is an incidence containing a_{h-1} and a_h. The *length of a weak walk* is half the number of incidences in the weak walk. The *sign of a weak walk* $\widetilde{W}$ is

$$sgn(\widetilde{W}) = (-1)^{\lfloor n/2 \rfloor} \prod_{h=1}^{n} \sigma(i_h).$$

A *vertex-walk* is a weak walk where a_0, $a_n \in V$, and $i_{2h-1} \neq i_{2h}$. This forbids the weak walk from entering an edge and immediately returning to the same vertex along the same incidence. A *vertex-path* is a vertex walk where no vertex or edge is repeated, while a *circle* is a vertex-path except $a_0 = a_n$.

The number of weak walks of length k from v_i to v_j is denoted $\widetilde{w}(v_i, v_j; k)$, the number of positive weak walks of length k is $\widetilde{w}^+(v_i, v_j; k)$, and the number of negative weak walks of length k is $\widetilde{w}^-(v_i, v_j; k)$, and let $\widetilde{w}^\pm(v_i, v_j; k) := \widetilde{w}^+(v_i, v_j; k) - \widetilde{w}^-(v_i, v_j; k)$. We use the analogous notation $w(v_i, v_j; k)$, $w^+(v_i, v_j; k)$, $w^-(v_i, v_j; k)$, and $w^\pm(v_i, v_j; k)$ to count walks. The concept of a weak walk was introduced in [8] to provide a unified combinatorial interpretation of the entries of the Laplacian of incidence-simple oriented hypergraphs.

2.2. Oriented hypergraphic matrices

Two, not necessarily distinct, vertices v and w are said to be *adjacent with respect to edge* e if there exist incidences (v, e, k_1) and (w, e, k_2) such that $(v, e, k_1) \neq (w, e, k_2)$. An *adjacency* is a quintuple $(v, k_1; w, k_2; e)$ where v and w are adjacent with respect to edge e using incidences (v, e, k_1) and (w, e, k_2).

Given an adjacency $(v, k_1; w, k_2; e)$ we define the *sign of the adjacency* as

$$sgn_e(v, k_1; w, k_2) = -\sigma(v, e, k_1)\sigma(w, e, k_2),$$

and we regard $sgn_e(v, k_1; w, k_2) = 0$ if v and w are not adjacent.

The *adjacency matrix* $A_G = [a_{ij}]$ of an oriented hypergraph G is the $V \times V$ matrix whose (i, j)-entry is the sum of all signed adjacencies containing vertices v_i and v_j. That is,

$$a_{ij} = \sum_{e;m,n} sgn_e(v_i, m; v_j, n) \tag{2.1}$$

where the sum is over all edges $e \in E$ and incidences $(v_i, e, m) \neq (v_j, e, n)$. Notice that two incidences (v_i, e, m) and (v_j, e, n) belonging to the adjacency $(v_i, m; v_j, n; e)$ also belong to the adjacency $(v_j, n; v_i, m; e)$, so clearly the adjacency matrix is symmetric. But most importantly, the diagonal entries of A_G count adjacencies of the form $(v_i, m; v_i, n; e)$ separate from $(v_i, n; v_i, m; e)$ when $m \neq n$; moreover, if $m = n$ the incidences are not in an adjacency.

The *degree of a vertex* v is the number of incidences containing vertex v. The *degree matrix* of an oriented hypergraph G is the diagonal matrix $D_G = [d_{ij}] := \mathrm{diag}(\deg(v_1), \ldots, \deg(v_n))$, while the *Laplacian matrix* is defined as $L_G := D_G - A_G$, and the *incidence matrix* $\mathrm{H}_G = [\eta_{ij}]$ is the $n \times m$ matrix defined by

$$\eta_{ij} = \sum_{k=1}^{\iota(v_i, e_j)} \sigma(v_i, e_j, k). \tag{2.2}$$

The following two theorems are strengthenings of the results in [8] by removing the condition of incidence-simplicity. The proofs are standard, but included for completeness.

Theorem 2.1. *If G is an oriented hypergraph and k a non-negative integer, then the (i,j)-entry of A_G^k is $w^{\pm}(v_i, v_j; k)$.*

Proof. We prove this using mathematical induction.

If $k = 0$, then $A_G^0 = I$, or a 0-walk travels nowhere.

If $k = 1$, then $\left(A_G^1\right)_{ij} = a_{ij}$, where

$$\begin{aligned} a_{ij} &= \sum_{e;m,n} sgn_e(v_i, m; v_j, n) \\ &= \sum_{e;m,n} -\sigma(v_i, e, m)\sigma(v_j, e, n) \\ &= w^+(v_i, v_j; 1) - w^-(v_i, v_j; 1) = w^{\pm}(v_i, v_j; 1). \end{aligned}$$

We now suppose that $\left(A_G^k\right)_{ij} = w^{\pm}(v_i, v_j; k)$.

We calculate the (i,j)-entry of A_G^{k+1} as follows:

$$\begin{aligned} \left(A_G^{k+1}\right)_{ij} &= \left(A_G A_G^k\right)_{ij} = \sum_{l=1}^{n} a_{il} \cdot w^{\pm}(v_l, v_j; k) \\ &= \sum_{l=1}^{n} \left(\sum_{e;m,n} sgn_e(v_i, m; v_l, n) \right) [w^+(v_l, v_j; k) - w^-(v_l, v_j; k)] \\ &= \sum_{l=1}^{n} (w^+(v_i, v_l; 1) - w^-(v_i, v_l; 1))[w^+(v_l, v_j; k) - w^-(v_l, v_j; k)] \\ &= \sum_{l=1}^{n} [w^+(v_i, v_l; 1)w^+(v_l, v_j; k) + w^-(v_i, v_l; 1)w^-(v_l, v_j; k) \\ &\quad - w^-(v_i, v_l; 1)w^+(v_l, v_j; k) - w^+(v_i, v_l; 1)w^-(v_l, v_j; k)]. \end{aligned}$$

The number of positive walks of length $k+1$ from v_i to v_j with v_l as the second vertex is

$$w^+(v_i, v_l; 1)w^+(v_l, v_j; k) + w^-(v_i, v_l; 1)w^-(v_l, v_j; k),$$

while the number of negative walks of length $k+1$ from v_i to v_j with v_l as the second vertex is

$$w^-(v_i, v_l; 1)w^+(v_l, v_j; k) + w^+(v_i, v_l; 1)w^-(v_l, v_j; k).$$

Since any walk of length $k+1$ from v_i to v_j must have one of the vertices $v_1, v_2, \ldots, v_n$ as its second vertex, we have

$$w^+(v_i, v_j; k+1) = \sum_{l=1}^{n} \left[w^+(v_i, v_l; 1)w^+(v_l, v_j; k) + w^-(v_i, v_l; 1)w^-(v_l, v_j; k)\right],$$

$$w^-(v_i, v_j; k+1) = \sum_{l=1}^{n} \left[w^-(v_i, v_l; 1)w^+(v_l, v_j; k) + w^+(v_i, v_l; 1)w^-(v_l, v_j; k)\right],$$

and the (i, j)-entry of A_G^{k+1} simplifies to

$$w^+(v_i, v_j; k+1) - w^-(v_i, v_j; k+1) = w^{\pm}(v_i, v_j; k+1).$$

Completing the proof. □

Theorem 2.2. *If G is an oriented hypergraph, then $L_G = \mathrm{H}_G \mathrm{H}_G^T$.*

Proof. Observe that the (i, j)-entry of $\mathrm{H}_G \mathrm{H}_G^T$ corresponds to the ith row of H_G multiplied by the jth column of H_G^T, where $v_i, v_j \in V$. Therefore, this entry is

$$\begin{aligned}
\sum_{l=1}^{|E|} \eta_{i,l}\eta_{j,l} &= \sum_{l=1}^{|E|} \Big[\sum_{M=1}^{\iota(v_i,e_l)} \sigma(v_i, e_l, M)\Big]\Big[\sum_{N=1}^{\iota(v_j,e_l)} \sigma(v_j, e_l, N)\Big] \\
&= \sum_{l=1}^{|E|} \Big[\sum_{M=1}^{\iota(v_i,e_l)} \sum_{N=1}^{\iota(v_j,e_l)} \sigma(v_i, e_l, M)\sigma(v_j, e_l, N)\Big] \\
&= \sum_{l=1}^{|E|} \Big[\sum_{M=1}^{\iota(v_i,e_l)} \sum_{N=1}^{\iota(v_j,e_l)} -sgn_{e_l}(v_i, M; v_j, N)\Big]
\end{aligned}$$

Case 1: If $i = j$ we break the sum into parts where $M = N$ and $M \neq N$.

$$\begin{aligned}
&= \sum_{l=1}^{|E|} \Big[\sum_{M=1}^{\iota(v_i,e_l)} \sum_{N=1}^{\iota(v_i,e_l)} \sigma(v_i, e_l, M)\sigma(v_i, e_l, N)\Big] \\
&= \sum_{l=1}^{|E|} \Big[\sum_{\substack{M=1,\\ N=M}}^{\iota(v_i,e_l)} (\sigma(v_i, e_l, M))^2 \Big] + \sum_{l=1}^{|E|} \Big[\sum_{\substack{M=1,\\ M\neq N}}^{\iota(v_i,e_l)} \sum_{N=1}^{\iota(v_i,e_l)} \sigma(v_i, e_l, M)\sigma(v_i, e_l, N)\Big] \\
&= \sum_{l=1}^{|E|} \Big[\sum_{\substack{M=1,\\ N=M}}^{\iota(v_i,e_l)} (\sigma(v_i, e_l, M))^2 \Big] + \sum_{l=1}^{|E|} \Big[\sum_{\substack{M=1,\\ M\neq N}}^{\iota(v_i,e_l)} \sum_{N=1}^{\iota(v_i,e_l)} -sgn_{e_l}(v_i, M; v_i, N)\Big] \\
&= \deg(v_i) - a_{ii}
\end{aligned}$$

Notice that if $M = N$ then the inner sums merge and $(\sigma(v_i, e_l, M))^2 = 1$, and we count the number of incidences containing v_i and e_l over all possible edges e_l, and get $\deg(v_i)$. However, if $M \neq N$ then we count all $-sgn_{e_l}(v_i, M; v_i, N)$ over all adjacencies from v_i to v_i within edge e_l and sum over all possible edges, and get $-a_{ii}$. Thus the (i,i)-entry of $\mathrm{H}_G\mathrm{H}_G^T$ is $\deg(v_i) - a_{ii}$.

Case 2: If $i \neq j$ the sum simplifies as

$$
\begin{aligned}
&= \sum_{l=1}^{|E|} \Big[\sum_{M=1}^{\iota(v_i,e_l)} \sum_{N=1}^{\iota(v_j,e_l)} \sigma(v_i,e_l,M)\sigma(v_j,e_l,N) \Big] \\
&= \sum_{l=1}^{|E|} \Big[\sum_{M=1}^{\iota(v_i,e_l)} \sum_{N=1}^{\iota(v_j,e_l)} -sgn_{e_l}(v_i,M;v_j,N) \Big] \\
&= -a_{ij}
\end{aligned}
$$

Thus the (i,j)-entry of $\mathrm{H}_G\mathrm{H}_G^T$ is $-a_{ij}$.

Comparing entries we have $\mathrm{H}_G\mathrm{H}_G^T = D_G - A_G = L_G$. $\square$

3. Weak walk matrices

Let $A_1, A_2 \in \{V, E\}$ and $\widetilde{W}_{(G,A_1,A_2,k)} = [\widetilde{w}_{ij}]$ be the $A_1 \times A_2$ matrix where $\widetilde{w}_{ij} = \widetilde{w}^{\pm}(a_i, a_j; k)$. The matrix $\widetilde{W}_{(G,A_1,A_2,k)}$ is called the *weak k-walk matrix* of an oriented hypergraph G. We will assume that $A_1 = A_2 = V$ and only consider weak vertex-walks. It was shown in [8] that weak 1-walks count the entries of the Laplacian, as given by the following theorem and lemma.

Theorem 3.1. *(See Reff and Rusnak [8].) If G is an oriented hypergraph, then $L_G = -\widetilde{W}_{(G,V,V,1)}$.*

We define the *k-walk matrix* $W_{(G,A_1,A_2,k)} = [w_{ij}]$ similar to the weak k-walk matrix except $w_{ij} = w^{\pm}(a_i, a_j; k)$ and we get the following immediate reformulation of Theorem 2.1.

Lemma 3.2. *(See Reff and Rusnak [8].) If G is an oriented hypergraph, then $W_{(G,V,V,k)} = A_G^k$.*

While signed walk counts and powers of the adjacency matrix are related by the previous lemma, we provide a new relationship linking signed weak walk counts to powers of the Laplacian matrix.

Theorem 3.3. *If G is an oriented hypergraph, then $\widetilde{W}^k_{(G,V,V,1)} = \widetilde{W}_{(G,V,V,k)} = (-1)^k L_G^k$.*

Proof. We prove the second equality via mathematical induction similar to Theorem 2.1. The first equality is similar.

If $k = 0$, then $L_G^0 = I$, or a weak 0-walk travels nowhere.

If $k = 1$, then $L_G = -\widetilde{W}_{(G,V,V,1)}$ from Theorem 3.1.

We now suppose that $\left(L_G^k\right)_{ij} = (-1)^k \cdot \widetilde{w}^{\pm}(v_i, v_j; k)$.

We calculate the (i, j)-entry of L_G^{k+1} as follows:

$$\begin{aligned}\left(L_G^{k+1}\right)_{ij} = \left(L_G L_G^k\right)_{ij} &= \sum_{l=1}^{n} (-1)^1 \cdot \widetilde{w}^{\pm}(v_i, v_l; 1) \cdot (-1)^k \cdot \widetilde{w}^{\pm}(v_l, v_j; k) \\ &= (-1)^{k+1} \sum_{l=1}^{n} [\widetilde{w}^{+}(v_i, v_l; 1)\widetilde{w}^{+}(v_l, v_j; k) + \widetilde{w}^{-}(v_i, v_l; 1)\widetilde{w}^{-}(v_l, v_j; k) \\ &\quad - \widetilde{w}^{-}(v_i, v_l; 1)\widetilde{w}^{+}(v_l, v_j; k) - \widetilde{w}^{+}(v_i, v_l; 1)\widetilde{w}^{-}(v_l, v_j; k)].\end{aligned}$$

The number of positive weak walks of length $k + 1$ from v_i to v_j with v_l as the second vertex is

$$\widetilde{w}^{+}(v_i, v_l; 1)\widetilde{w}^{+}(v_l, v_j; k) + \widetilde{w}^{-}(v_i, v_l; 1)\widetilde{w}^{-}(v_l, v_j; k),$$

while the number of negative weak walks of length $k + 1$ from v_i to v_j with v_l as the second vertex is

$$\widetilde{w}^{-}(v_i, v_l; 1)\widetilde{w}^{+}(v_l, v_j; k) + \widetilde{w}^{+}(v_i, v_l; 1)\widetilde{w}^{-}(v_l, v_j; k).$$

Summing these positive and negative weak walks over all possible second vertices we have:

$$\widetilde{w}^{+}(v_i, v_j; k+1) = \sum_{l=1}^{n} \left[\widetilde{w}^{+}(v_i, v_l; 1)\widetilde{w}^{+}(v_l, v_j; k) + \widetilde{w}^{-}(v_i, v_l; 1)\widetilde{w}^{-}(v_l, v_j; k)\right],$$

and

$$\widetilde{w}^{-}(v_i, v_j; k+1) = \sum_{l=1}^{n} \left[\widetilde{w}^{-}(v_i, v_l; 1)\widetilde{w}^{+}(v_l, v_j; k) + \widetilde{w}^{+}(v_i, v_l; 1)\widetilde{w}^{-}(v_l, v_j; k)\right].$$

Thus, the (i, j)-entry of L_G^{k+1} simplifies to

$$(-1)^{k+1} \cdot [\widetilde{w}^{+}(v_i, v_j; k+1) - \widetilde{w}^{-}(v_i, v_j; k+1)] = (-1)^{k+1} \cdot \widetilde{w}^{\pm}(v_i, v_j; k+1).$$

Completing the proof. □

Relating weak walks back to the incidence matrix we have the following known result:

Lemma 3.4. *(See Reff and Rusnak [8].) If G is an oriented hypergraph, then* $\widetilde{W}_{(G,V,E,1/2)} = \mathrm{H}_G$.

The importance of Theorem 3.3 is when coupled with the results from [8] we see that the entries of the degree, adjacency, incidence, and Laplacian matrices can all be interpreted using weak walks.

4. Balance and a Harary-type theorem

4.1. Definitions and background

Recall that an oriented hypergraph is *balanced* if the sign of all circles are positive. Harary's theorem tells us that a signed graph is balanced if, and only if, all vw-paths have the same sign. However, Harary's theorem fails immediately if we allow edges of size 3 or greater, as indicated in the following figure:

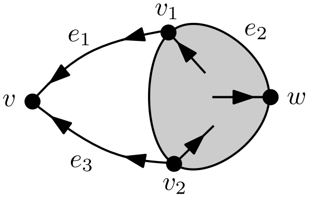

Fig. 1. A balanced oriented hypergraph where the two vw-paths have different signs.

In Fig. 1 we see that the oriented hypergraph is balanced as the only circle $v, e_1, v_1, e_2, v_2, e_3, v$ is positive. However, the vw-paths $P_1 = v, e_1, v_1, e_2, w$ and $P_2 = v, e_3, v_2, e_2, w$ have $sgn(P_1) = +1$ and $sgn(P_2) = -1$. Clearly there is an issue using the extra incidence between e_2 and w as it is reused when connecting the two paths. To incorporate this phenomenon into the Harary's theorem we need to examine weak walks, but before we do this we need to collect some definitions and known results from [9] for balancing obstructions in oriented hypergraphs.

The *oriented incidence graph of an oriented hypergraph* $G = (V_G, E_G, \mathcal{I}_G, \sigma)$ is the oriented bipartite graph Γ_G with vertex set $V_\Gamma = V_G \cup E_G$, edge set $E_\Gamma = \mathcal{I}_G$ and orientation function σ.

Lemma 4.1. *(See Rusnak [9].)* $\widetilde{W}$ *is a weak walk of* G *if, and only if,* $\widetilde{W}$ *is a walk in* Γ_G*.*

A *theta graph* is a set of three internally disjoint paths with the same end-points. A *vertex-theta-graph* is a theta graph whose end-points are vertices, an *edge-theta-graph* is a theta graph whose end-points are edges, and a *cross-theta-graph* is a theta graph whose end-points consist of one vertex and one edge. This is shown in Fig. 2.

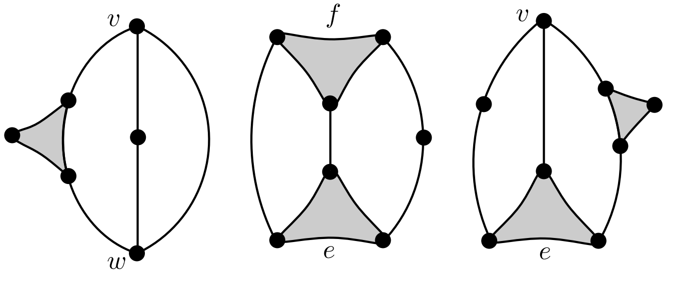

Fig. 2. Hypergraphs that contain a vertex-theta, edge-theta, and cross-theta, respectively.

Clearly, the paths of a theta graph form three internally disjoint paths in the bipartite incidence graph and we have the following result.

Lemma 4.2. *G is cross-theta-free if, and only if, there are no chordal paths of odd length in Γ_G.*

An oriented hypergraph is *balanceable* if there are incidences that can be negated so that the resulting oriented hypergraph is balanced, while an oriented hypergraph that is not balanceable is said to be *unbalanceable*. We know the following:

Theorem 4.3. *(See Rusnak [9].) An oriented hypergraph G is balanceable if, and only if, it does not contain a cross-theta.*

Observe that the underlying hypergraphic structure of balanced and balanceable oriented hypergraphs are identical and they only differ by incidence orientations, so a balanced oriented hypergraph is cross-theta-free.

4.2. Self-intersection and defect

Given weak walk $\widetilde{W}$ define the *self-multiplicity of incidence i in $\widetilde{W}$*, denoted $m(i;\widetilde{W})$, as the number of times incidence i appears in weak walk $\widetilde{W}$. Note that self-multiplicity is not oriented hypergraphic incidence multiplicity, as oriented hypergraphs allow multiple distinct incidences of the form (v,e,k_1) and (v,e,k_2) that are only equal only when $k_1 = k_2$. Self-multiplicity counts the number of times a specific incidence $i = (v,e,k)$ appears in a weak walk.

Define the *self-intersection number of weak walk $\widetilde{W}$* as

$$\gamma_{\widetilde{W}} := \sum_{i \in \mathcal{I}_{\widetilde{W}}} \left\lfloor \frac{m(i;\widetilde{W})}{2} \right\rfloor .$$

The self-intersection number increases by 1 for every two times an incidence occurs in a weak walk. The *defect of weak walk $\widetilde{W}$* is defined as

$$\delta_{\widetilde{W}} := (-1)^{\gamma_{\widetilde{W}}}.$$

Theorem 4.4. *The sign of a closed weak walk in a balanced oriented hypergraph is its defect $\delta_{\widetilde{W}}$.*

Proof. Let G be a balanced oriented hypergraph whose bipartite incidence graph is Γ, and let $\widetilde{W}$ be a closed weak walk in G.

Observe that if any subset of $\widetilde{W}$ forms a circle C then $sgn(\widetilde{W}) = sgn(\widetilde{W} \smallsetminus C)$ since G is balanced and the sign of every circle is positive. Thus, we may sequentially remove circles from $\widetilde{W}$ resulting in a collection of circle-free weak walks X where $sgn(\widetilde{W}) = sgn(X)$.

We know from Lemma 4.1 that $\widetilde{W}$ is a closed walk in the bipartite incidence graph Γ, so each incidence of X (an edge in Γ) must appear an even number of times in X. Let these multiplicities be $2\alpha_1, 2\alpha, \ldots, 2\alpha_k$, and let $\alpha := \alpha_1 + \alpha_2 + \ldots + \alpha_k$. From the definition we have $sgn(\widetilde{W}) = sgn(X) = (-1)^{\alpha}$.

However, the self-intersection number γ counts all repeated incidence pairs of $\widetilde{W}$ so we now examine the $\gamma - \alpha$ uncounted repeated incidence pairs of $\widetilde{W}$ belonging to the removed circles. Regard these circles as circles in the bipartite incidence graph Γ, so each incidence in G is an edge in Γ. When two circles C_1 and C_2 are removed that contain common incidences either $C_1 = C_2$, or the incidences of $C_2 \setminus C_1$ can be regarded as the edges of chordal paths to C_1 in Γ. If $C_1 = C_2$ then they each contain an even number of incidences since Γ is bipartite. However, if the circles are distinct we know from Lemma 4.2 that all chordal paths have even length in Γ, and since Γ is bipartite and C_2 has even length $C_1 \cap C_2$ must consist of an even number of edges in Γ, that is, $C_1 \cap C_2$ contain an even number of incidences in G.

Thus, $\gamma - \alpha$ is even and $\alpha \equiv \gamma \bmod 2$, so $(-1)^{\alpha} = (-1)^{\gamma}$, and the theorem is proved. □

Using Theorem 4.4 along with the observation that a circle has a self-intersection of 0 and has a defect $+1$ we have the following theorem:

Theorem 4.5. *An oriented hypergraph is balanced if, and only if, the sign of each closed weak walk is equal to its defect.*

This gives us Harary's Theorem as a corollary.

Corollary 4.6. *A signed graph is balanced if, and only if, for each pair of vertices v and w all vw-paths have the same sign.*

Proof. Let P_1 and P_2 be two paths between vertices v_1 and v_2 in a balanced signed graph G. Regard $P_1 \cup P_2$ as a closed (non-weak) walk W since P_1 and P_2 are paths and the only edges appearing in a path in a signed graph are 2-edges. Observe that every incidence i that appears in a path must include the other unique incidence i' belonging to their common 2-edge, so if an incidence i appears in both P_1 and P_2 so must i'. Thus, γ_W is even, $\delta_W = +1$, and

$$sgn(P_1) \cdot sgn(P_2) = sgn(W) = \delta_W = +1.$$

So, $sgn(P_1) = sgn(P_2)$.

To see the converse, decompose every circle into two internally-disjoint vw-paths for some v and w in each circle and use the assumption that for each pair of vertices v and w all vw-paths have the same sign, so each circle must be positive, and G is balanced. □

Returning to Fig. 1 we know that paths $P_1 = v, e_1, v_1, e_2, w$ and $P_2 = v, e_3, v_2, e_2, w$ combine to form a negative closed weak walk $\widetilde{W} = v, e_1, v_1, e_2, w, e_2, v_2, e_3, v$; incidences are omitted since the hypergraph is incidence-simple. This weak walk contains the incidences of circle v, e_1, v_1, e_2, v_2, e_3, v which is positive, as well as the repeated incidence between w and e_2. Since there is one repeated incidence we have $\gamma_{\widetilde{W}} = 1$ and the sign of $\widetilde{W}$ is $(-1)^1$.

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