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ARITHMETIC AND GEOMETRIC MEAN

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Consider the following problem:

Among all rectangles with a given perimeter, find the rectangle with the largest area?

If we draw a rectangle, we can label the sides as x_1 and x_2. The perimeter P is then $P = 2(x_1 + x_2)$. With a little experimentation, we might guess that the rectangle with largest area occurs when $x_1 = x_2 = \frac{P}{4}$, i.e. that the answer is a square. How can we prove this? Algebraically, we must show that for any choice of variables x_1 and x_2, subject to the condition $P = 2(x_1+x_2)$, we will have condition (1) below:

$$x_1 x_2 \leq \left(\frac{P}{4}\right)^2. \tag{1}$$

Substituting, we obtain equation (2)

$$x_1 x_2 \leq \left(\frac{2(x_1+x_2)}{4}\right)^2. \tag{2}$$

Simplifying, this becomes

$$\begin{aligned} 4x_1x_2 &\leq (x_1+x_2)^2 \\ 0 &\leq x_1^2 - 2x_1x_2 + x_2^2 \\ 0 &\leq (x_1-x_2)^2 \end{aligned}$$

which is true! Note that we have equality if and only if $x_1 = x_2$. Equation (2) above can be rewritten as (3)

$$\sqrt{x_1x_2} \leq \frac{(x_1+x_2)}{2}. \tag{3}$$

We call the quantity on the left the geometric mean, G, of x_1 and x_2, and the quantity on the right the arithmetic mean, M. In words, we have proved that the geometric mean G of two numbers x_1, x_2 is always less than or equal to the arithmetic mean M with equality if and only if $x_1 = x_2$.

A slightly different proof of this important fact can be given as follows. Fix a quantity P, and pick any two numbers x_1 and x_2 satisfying $P = 2(x_1 + x_2)$. We now want to make the quantity $x_1 x_2$ as large as possible.

Now let's think about the arithmetic mean, $M = \dfrac{x_1 + x_2}{2} = \dfrac{P}{4}$. Geometrically, M is the midpoint of x_1 and x_2. We let d be the distance from the midpoint to either x_1 or x_2. Assume $x_1 > x_2$, so $d = \dfrac{(x_1 - x_2)}{2}$. Observe that

$$x_1 = M + d\,,$$
$$x_2 = M - d\,.$$

Hence

$$x_1 x_2 = (M + d)(M - d) = M^2 - d^2 = \left(\frac{P}{4}\right)^2 - d^2\,.$$

Observe that the right hand side of this equation is the difference of two squares, and the conditions of our problem state that P is constant. Hence, the largest possible value of $x_1 x_2$ will occur when $d = 0$, i.e. when $x_1 = x_2$. We write this as an inequality:

$$x_1 x_2 = \left(\frac{P}{4}\right)^2 - d^2 \le \left(\frac{P}{4}\right)^2\,.$$

Taking square roots, we have

$$G = \sqrt{x_1 x_2} \le \frac{P}{4} = M\,.$$

As an exercise, let's use the above method to generalize these ideas to more than two quantities. We define the arithmetic mean, M, of quantities $x_1, x_2, \ldots, x_n$ by

$$M = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

and the geometric mean, G , by

$$G = \sqrt[n]{x_1 x_2 \ldots x_n}$$

where we assume that all the quantities are positive.

The analogous problem is to fix the sum $S = x_1 + x_2 + \cdots + x_n$, and try to find the maximum value of the product $Q = x_1 x_2 \ldots x_n$. We want to show that the maximum value of Q occurs when $x_1 = x_2 = \cdots = x_n$. Now observe that in this case, the geometric mean G is equal to the arithmetic mean M. In other words, the maximum value of the geometric mean will always be less than the arithmetic mean, $\frac{S}{n}$, with equality only when all the variables are equal.

So suppose that $x_1, x_2, \ldots, x_n$ give the maximum value of Q, and that two of the values are not equal, say $x_1 \neq x_2$. Proceeding as above, we can let

$$M = \frac{x_1 + x_2}{2},$$
$$d = \frac{x_1 - x_2}{2}.$$

Again, we have

$$x_1 = M + d,$$
$$x_2 = M - d.$$

We now choose a new set of quantities

$$M, M, x_3, x_4, \ldots, x_n.$$

This new set has the same sum S as before. However, the new product Q' will be

$$Q' = M^2 x_3 x_4 \ldots x_n.$$

This is clearly larger than the old product Q, where

$$\begin{aligned} Q &= x_1 x_2 x_3 \ldots x_n \\ &= (M+d)(M-d) x_3 x_4 \ldots x_n \\ &= (M^2 - d^2) x_3 x_4 \ldots x_n. \end{aligned}$$

We have equality if and only if $d = 0$, i.e. if $x_1 = x_2$. The same argument shows that all of the quantities, x_i, must be equal to obtain the maximum value.

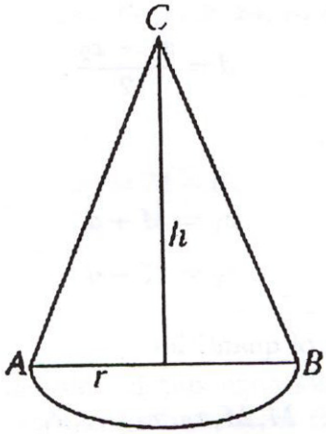
Let's now see how we can use the inequality $G \leq M$ in some problems. Before we begin, let's warm up with an application:

Arithmetic Lemma. *Prove that if a and b are positive integers satisfying $a + b = 1$, then $\frac{1}{ab} \geq 4$.*

Proof. Since $G \leq M$, $\sqrt{ab} \leq \frac{a+b}{2} = \frac{1}{2}$. Squaring both sides, then transposing, the result follows.

We will now use this in the geometry problem that follows.

Geometry Problem: Consider a circular cone, with base radius r, and height h. Let A and B be the endpoints of a diameter on the base of the cone, and C be the vertex of the cone. We let O be the center of the inscribed circle in triangle ABC, and let R be the radius of the inscribed circle. Let V be the volume of the cone, so $V = \frac{1}{3}\pi r^2 h$. We now want to show that $V \geq \frac{8}{3}\pi R^3$.



We let M be the midpoint of AB, so M is the midpoint of the diagonal AB on the base of the cone. We let T be the point where the inscribed circle intersects line AC. So triangle OTC and triangle AMC are similar. Thus, $\frac{R}{h-R} = \frac{r}{\sqrt{h^2+r^2}}$, and solving for r^2, we obtain $r^2 = \frac{hR^2}{h-2R}$. Substituting this into our equation for V, we find that

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h^2R^2}{h-2R}\right)$$

$$= \frac{1}{3}\pi\left(\frac{R^2}{\left(\frac{h-2R}{h^2}\right)}\right) = \frac{1}{3}\pi\left(\frac{R^2}{\frac{1}{h}\left(1-\frac{2R}{h}\right)}\right)$$

$$= \left(\frac{2R}{2R}\right)\frac{1}{3}\pi\left(\frac{R^2}{\frac{1}{h}\left(1-\frac{2R}{h}\right)}\right)$$

$$= \frac{1}{3}\pi\left(\frac{2R^3}{\frac{2R}{h}\left(1-\frac{2R}{h}\right)}\right)$$

By the Arithmetic Lemma, it follows that

$$V = \frac{1}{3}\pi 2R^3 \left(\frac{1}{\frac{2R}{h}\left(1 - \frac{2R}{h}\right)} \right) \geq \left(\frac{2}{3}\pi R^3\right)(4) = \frac{8}{3}\pi R^3 . \quad \text{Q.E.D.}$$

As a second example, consider the problem below:

What is the minimum value of $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$?

Using the fact that $M \geq G$, we obtain $a+b+c \geq 3\sqrt[3]{abc}$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3\sqrt[3]{\frac{1}{a}\frac{1}{b}\frac{1}{c}}$.

Multiplying these together,

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 3\sqrt[3]{abc}\, 3\sqrt[3]{\frac{1}{abc}} = 9$$

with equality only when $a = b = c$.

The arithmetic mean also has an interesting minimum property. Namely, if we make measurements $x_1, x_2, \ldots, x_n$, the question is what is the actual best value we should accept for the quantity we are trying to measure? One way to look at this problem, following Gauss [1], [2, p. 365], is to try to determine a quantity Q with the property that the sum of the square of the deviations $(Q-x_1)^2 + (Q-x_2)^2 + \cdots + (Q-x_n)^2$ is as small as possible. We claim that then the optimal value of Q is the arithmetic mean. To prove this, write $Q - x_i = (M - x_i) + (Q - M)$ where Q is any value we accept for our measurement, and M is the arithmetic mean of the observed values. Then

$$(Q-x_i)^2 = (M-x_i)^2 + (Q-M)^2 + 2(M-x_i)(Q-M) .$$

Summing these equations, we note that the last terms sum to 0, because $2(Q-M)(nM - x_1 - x_2 - \cdots - x_n) = 0$ by definition of M.
Hence the sum of these equations becomes

$$\begin{aligned} &(Q-x_1)^2 + (Q-x_2)^2 + \cdots + (Q-x_n)^2 \\ &= (M-x_1)^2 + \cdots + (M-x_n)^2 + n(Q-M)^2 . \end{aligned}$$

This shows that

$$(Q-x_1)^2 + (Q-x_2)^2 + \cdots + (Q-x_n)^2 \geq (M-x_1)^2 + \cdots + (M-x_n)^2$$

with equality if and only if $Q = M$.

References

1. Louise S. Grinstein and Sally I. Lipsey, editors, *Least Squares* in *Encyclopedia of Mathematics Education*, RoutledgeFalmer, New York, 2001, p. 403.
2. Richard Courant and Herbert Robbins, *What is Mathematics?*, Oxford University Press, New York, 1941.

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