Combinatorial Applications of Grassmann Algebra on Laplacian Matrices and Subgraph Enumerations

Abstract

In the field of graph theory, Laplacian matrices corresponding to graphs prove useful in deriving graph properties. Grassmann algebra and the Berezin integral, devised for integrating fermionic fields, possess combinatorial properties via exponential generating functions. In this paper, we integrate Grassmann polynomials to formulate expressions for the determinant and permanent of Laplacian matrices. After deriving an alternate proof of Kirchhoff's Theorem, we introduce modifications to the Grassmann algebra; this allows us to evaluate the Laplacian permanent and construct an alternative proof for its enumeration of bipartite subgraphs. We then generalize this enumeration to submatrices and modify the Laplacian to generalize enumerated subgraphs to an even greater extent. These computations possess applications in graphical situations involving high levels of connectivity; specifically, the bijection between even-cycled graphs and bipartite graphs is key for specific data structures. Our results prove especially ideal in formulating an efficient method in computing paths and cycles within complex biological interaction graphs. Our paper presents a rigorous set of applications for Grassmann algebra in order to derive properties of Laplacian permanents, which are widely believed to be too difficult to calculate and impractical to use, and to count specific and applicable subgraphs.

1 Introduction

Graph theory has long served as the premier tool in modeling networks and connections between object pairs. Graph structures are pervasively used in optimizing computer, electrical, and transportation networks, contributing to a rich understanding of graphical properties, and fulfilling useful applications to other mathematical and scientific fields.

Grassmann algebra has largely been applied to studying subatomic particles with odd half integer spin, known as fermions, in the field of physics [3]. However, Grassmann algebra has less known, but just as pertinent, applications in mathematics, specifically in linear algebra and combinatorics [7]. Matrix representations of graphs encode information about their edge sets and connectivity, and approaching these matrices using Grassmann algebra and Berezin integrals can derive additional information about graphs. For example, Grassmann algebra enables an algebraic proof for Kirchhoff's Theorem, which enumerates spanning trees in a graph via the determinant minor of the Laplacian matrix [1]. Similarly, Grassmann algebra has been applied to proving the Lindström-Gessel-Viennot Lemma, which relates the determinant of the matrix of a weighted graph to its lattice paths [5]. Grassmann algebra is an extremely useful tool in generating determinants and permanents of graph matrices and thus holds a high level of potential in deriving new theorems in graph theory.

While Kirchhoff's Theorem employs the determinant of the graph Laplacian, another characteristic of the matrix that can be calculated from the entries is the permanent. There has been extensive and ongoing research regarding how to calculate the permanent of a matrix in polynomial time, as can be done for the determinant. Because of the complexity of its computation, permanents are not as widely studied as determinants. Similar to how Kirchhoff's Theorem enumerates spanning trees in a graph from its matrix representation, our results show that the permanent of Laplacian matrices is able to count several types of graph features.

2 Background Information

2.1 Definitions

Definition 2.1. A *graph* is an ordered pair $G = (V, E)$ in which V is a set of vertices (denoted $\{v_1, v_2, \ldots, v_n\}$) and edge set E is a set of two-element subsets of V. Visual representations of graphs use points to denote vertices and lines connecting points to denote edges.

Definition 2.2. The *degree* of vertex v_i, or $deg(v_i)$, is the number of edges connected to v_i.

Definition 2.3. A *directed edge* is an ordered pair of vertices (v_i, v_j) such that v_i is the base of the edge and v_j is the destination of the edge. The direction is from v_i to v_j. For directed edges, (v_i, v_j) and (v_j, v_i) are distinct and may both exist in a directed graph.

Remark. For directed graphs (defined below), each vertex has an indegree and an outdegree. The *indegree* of vertex v_i, denoted $indeg(v_i)$, is equal to the number of vertices v_j such that directed edge (v_j, v_i) exists. The *outdegree* of vertex v_i, denoted $outdeg(v_i)$, is equal to the number of vertices v_j such that directed edge (v_i, v_j) exists.

Definition 2.4. A *directed graph* is an ordered pair $G = (V, A)$ in which V is a set of vertices and A is a set of directed edges. Visually, an arrow is drawn for each directed edge, with the head representing the destination and the tail representing the base.

Definition 2.5. A *subgraph* of a graph $G = (V, E)$ is a graph $H = (V', E')$ such that V' and E' are subsets of V and E respectively, and E' is a set containing two-element subsets of V'.

Remark. In our research, we exclusively consider subgraphs where $V' = V$.

Definition 2.6. For a graph with n vertices, the *Laplacian matrix* of a graph L, also known

as the *graph Laplacian*, is an $n \times n$ matrix where

$$L_{ij} = \begin{cases} deg(\mathrm{v}_i) & \text{if } i = j \\ -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of a vertex degree, the sum of the terms in any row or column is zero.

Definition 2.7. A *submatrix* M_{IJ} of matrix M is a matrix formed by deleting a set of rows I and a set of rows J from M.

Definition 2.8. A *path* is a sequence of vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ in which vertices adjacent in the sequence are connected by an edge. The number $k-1$ denotes the *length* of the path.

Definition 2.9. A *cycle* is a sequence of vertices $\{v_1, v_2, \ldots, v_k\}$ in which vertices adjacent in the sequence are connected by an edge, and there is an edge connecting (v_1, v_k). A cycle is considered *even* when k is even and *odd* when k is odd. In this case, k denotes the *length* of the cycle.

Remark. Only directed graphs can have cycles with length 2.

Definition 2.10. A *bipartite graph* (also known as a bigraph or a two-colorable graph) is a graph whose vertex set can be partitioned into two sets of vertices such that no two vertices in the same set are adjacent. It can be shown that a graph is bipartite if and only if it contains no odd cycles.

Definition 2.11. A *connected component* (or *component*) of a graph G is a subset of vertices S such that any pair of vertices in S is connected by at least one path, and any vertex not in S has no paths connecting it to any vertex in S.

Definition 2.12. A *spanning tree* of a graph with n vertices is a subgraph with $n-1$ edges such that between each pair of vertices a path exists. It follows that spanning trees do not contain any cycles.

Definition 2.13. The *determinant* of a square matrix A is denoted as $\det(A)$ or $|A|$. The formula for the determinant is given as $\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$ where σ is a permutation of the set $\{1, 2, 3, \ldots, n\}$ and $sgn(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$

Definition 2.14. The *permanent* of square matrix A, denoted as $\text{perm}(A)$, represents the unsigned summation of permutation products. The explicit formula for an $n \times n$ matrix is

$$\text{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Definition 2.15. A *minor* M_{IJ} of matrix M is the determinant of a square submatrix with row set I and column set J deleted, where $|I| = |J|$.

Definition 2.16. The summation notation $\sum_{i,j}^{n}$ indicates the summation over all pairs of positive integers $i, j \leq n$. The definition is analogous for $\prod_{i,j}^{n}$.

2.2 Grassmann Algebra

Grassmann algebra is a type of algebra on Grassmann variables which, when expressed in products or used in polynomials, possesses useful mathematical properties. With generating functions and the Berezin integral, it may be used to enumerate matrix determinants and permanents. Specifically, we can use Grassmann variables in the context of vertices on a graph and deduce permanents and determinants in Laplacian matrices.

Definition 2.17. *Grassmann variables* are anticommutative variables that are usually written as $\chi_1, \chi_2, \ldots, \chi_n$ for $n \in \mathbb{N}$:

$$\chi_i \chi_j = -\chi_j \chi_i, \quad \forall i, j \in 1, 2, \ldots, n.$$

$$\therefore \chi_i^2 = 0, \quad \forall i \in 1, 2, \ldots, n.$$

We define a function in Grassmann algebra to be a function f such that

$$f(\chi) = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n} a_{i_1} \ldots a_{i_k} \chi_{i_1} \ldots \chi_{i_k},$$

where each a_i is a numerical constant. The equation implies the function f is made up of nonzero monomials in Grassmann variables, where each term has a coefficient ("weight").

By the Taylor series expansion of the exponential function (denoted as e^x or $\exp x$) function, we can conclude that for any Grassmann function f with n different variables,

$$\exp f(\chi) = e^{f(\chi)} = \sum_{l=0}^{+\infty} \frac{1}{l!} f(\chi)^l$$

This expression becomes a regular polynomial in terms of Grassmann variables. It is easily shown that if $f(\chi)$ is in the form $A\chi_1\chi_2 ... \chi_n$, then $f(\chi)^p = 0$ for $p > n$. Thus, the infinite summation produces a polynomial with finite degree.

Theorem 2.1. *Let f be a nonzero, non-constant summand of a Grassmann polynomial. Then we can conclude [5] that* $\exp f = 1 + f$.

Theorem 2.2. *Let g and h be summands of Grassmann polynomials. Then $e^g e^h = e^{g+h}$ if at least one of g or h is even, i.e. the degree of at least one of the summands is even [1].*

Definition 2.18. The *Berezin integral* on the Grassmann algebra is an extension of the path integral (not the traditional Lebesgue-style integral) where

$$\int d\chi_1 d\chi_2 \ldots d\chi_n \; \chi_1 \chi_2 \ldots \chi_n = 1$$

$$\int d\chi_1 d\chi_2 \ldots d\chi_n \; a\chi_1 \chi_2 \ldots \chi_n = a$$

$$\int d\chi_1 \; a = 0.$$

We may interpret the Berezin integral as an operation similar to partial derivatives. If $d\chi_i$ is the right-most differential, then the "coefficient" of χ_i is taken (which can include other variables) [9]. This process continues until every differential is evaluated, returning the integral value (the final coefficient).

For the purposes of this paper, we will always be examining constant integrals, i.e. integrals without any residual variables. Thus, we are specifically differentiating with respect to the Grassmann variables representing graph models.

2.3 Previous Work

Carrozza, Krajewski, and Tanasa [5] introduce two sets of Grassmann variables: the set $\chi = \{\chi_1, \chi_2, \ldots, \chi_n\}$ and the set $\bar{\chi} = \{\bar{\chi_1}, \bar{\chi_2}, \ldots, \bar{\chi_n}\}$. An $n \times n$ matrix A is also introduced. The following theorem is stated:

Theorem 2.3. *Using the notation* $d\bar{\chi}d\chi \stackrel{def}{=} d\bar{\chi_n}d\chi_n d\bar{\chi_{n-1}}d\chi_{n-1}\ldots d\bar{\chi_1}d\chi_1$,

$$|A| = \int d\bar{\chi}d\chi \ \exp(-\bar{\chi}A\chi) = \int d\bar{\chi}d\chi \ \exp\left(\sum_{i,j}^{n} \bar{\chi_i}A_{ij}\chi_j\right).$$

Theorem 2.3 sets a key precedent in generalizing Kirchhoff's Matrix Tree Theorem, a well-known theorem relating the enumeration of subgraphs to certain Laplacian minors and Hyperpfaffian graphs [1]. The concept of using the exponential generating function will also prove quite useful in our research, as the expanded form of the expression is fruitful in relating Grassmann variables to the calculation of permanents.

Theorem 2.4 (Kirchhoff's Theorem)**.** *Let G be a graph. The determinant of its graph Laplacian L with any row and its corresponding column deleted enumerates the number of spanning trees in G.*

Proof. Without loss of generality, we can consider L_{11} (the graph Laplacian with row 1 and column 1 removed). Note that the determinant of a minor is equal to the integral of the whole determinant in which χ_1 and $\bar{\chi}_1$ are not evaluated as differentials. Thus we have

$$|L_{11}| = \int d\bar{\chi}d\chi \ \chi_1\bar{\chi}_1 \exp(-\bar{\chi}L\chi) = \int d\bar{\chi}d\chi \ \chi_1\bar{\chi}_1 \exp\left(-\sum_{i,j}^{n} \bar{\chi_i}L_{ij}\chi_j\right),$$

where

$$\bar{\chi}L\chi = \sum_{j=1}^{n} \bar{\chi_j} \left(\sum_{i=1}^{n} L_{ij} \right) \chi_j + \sum_{i,j}^{n} (\bar{\chi_i} - \bar{\chi_j}) \, L_{ij}\chi_j.$$

Let $B_j \stackrel{\text{def}}{=} \sum_{i=1}^{n} L_{ij}$ for $1 \leq j \leq n$. Note that row sums and column sums are equal to zero for Laplacian matrices, so $B_j = 0$, and our expression becomes

$$\bar{\chi}L\chi = \sum_{i,j}^{n} (\bar{\chi_i} - \bar{\chi_j}) \, L_{ij}\chi_j.$$

Note that whenever $i = j$, the term in the summation becomes 0.

Using Taylor series expansion yields

$$|L_{11}| = \int d\bar{\chi}d\chi \; \chi_1\bar{\chi}_1 \exp(-\bar{\chi}L\chi) = \int d\bar{\chi}d\chi \; \chi_1\bar{\chi}_1 \exp\left(\sum_{i,j}^{n} - (\bar{\chi_i} - \bar{\chi_j}) \, L_{ij}\chi_j \right)$$

$$= \int d\bar{\chi}d\chi \; \chi_1\bar{\chi}_1 \prod_{i,j}^{n} \exp\left(- (\bar{\chi_i} - \bar{\chi_j}) \, L_{ij}\chi_j \right)$$

$$= \int d\bar{\chi}d\chi \; \chi_1\bar{\chi}_1 \prod_{i,j}^{n} \left(1 - (\bar{\chi_i} - \bar{\chi_j}) \, L_{ij}\chi_j \right).$$

Consider all such pairs i, j in the expression. If $i = j$, the factor is equal to 1 (since $\bar{\chi_i} - \bar{\chi_j} = 0$) and is therefore irrelevant in the product. If v_i and v_j are not connected by an edge, then $L_{ij} = 0$ and the factor is equal to 1. Thus, we can rewrite the determinant expression as

$$|L_{11}| = \int d\bar{\chi}d\chi \; \chi_1\bar{\chi}_1 \prod_{i,j|(v_i,v_j)\in E}^{n} \left(1 + (\bar{\chi_i} - \bar{\chi_j}) \, \chi_j \right).$$

Note that in this product of binomials, the integral evaluates the coefficient of $\chi_2\chi_3 \ldots \chi_n\bar{\chi}_2\bar{\chi}_3 \ldots \bar{\chi}_n$. This term is produced from a product of $n - 1$ factors in the form $(\bar{\chi_j} - \bar{\chi_i}) \, L_{ij}\chi_j$, where χ_j is distinct in each factor. Let χ_j represent v_j in graph G, and let $(\bar{\chi_j} - \bar{\chi_i})$ represent a directed edge from v_i to v_j.

Lemma 2.1. For Grassmann variables $\chi_1, \chi_2, \ldots, \chi_k$,

$$(\chi_2 - \chi_1)(\chi_3 - \chi_2) \ldots (\chi_k - \chi_{k-1})(\chi_1 - \chi_k) = 0.$$

Proof. In order to obtain a nonzero expression with χ_2 in the first factor (as a result of the fact that $\chi_i^n = 0$), we must combine χ_2 with χ_3 from the second factor, χ_4 from the third, and so on, yielding a term of $\chi_2\chi_3 \dots \chi_n\chi_1$, which is equal to $\chi_1\chi_2 \dots \chi_n(-1)^{n-1}$ by anticommutativity. Similarly, we have that χ_1 in the first factor only produces $(-\chi_1)(-\chi_2)\dots(-\chi_n) = \chi_1\chi_2 \dots \chi_n(-1)^n$. Since the two terms are additive inverses, we achieve the desired result.

Therefore, $|L_{11}|$ enumerates directed graphs with $n-1$ edges and no cycles. This is equal to the number of spanning trees in G, thus concluding the proof.

■

3 New Results

Apart from Bapat's theorem published in 1986, which we will prove and generalize using Grassmann algebra and Berezin integration, there has been little research on evaluating the permanent of graph Laplacians. Our work serves to extend the combinatorial aspects of the permanent (of both the graph Laplacian and its submatrices) and to carve out its place as an applicable matrix property in mathematics.

3.1 Modified Grassmann Algebra

To make Grassmann algebra applicable to expressing the permanent, we modify it as such:

Definition 3.1. *Modified Grassmann variables* are variables that are denoted as $\psi_1, \psi_2, \dots, \psi_n$ for $n \in \mathbb{N}$ and have the following properties:

$$\psi_i\psi_j = \psi_j\psi_i, \quad \forall i, j \in 1, 2, \dots, n$$

$$\psi_i^2 = 0 \quad \forall i \in 1, 2, \dots, n.$$

The key difference is that the variables of Modified Grassmann algebra (henceforth referred to as *MGA*) are commutative.

We will proceed to verify a Grassmann expression of a matrix permanent. The proof will be similar to the proof of Theorem 2.3 given by Carrozza *et al.*, but since it is an undocumented result, it will be provided here.

Theorem 3.1. *If A is an $n \times n$ matrix, then*

$$perm(A) = \int d\bar{\psi} d\psi \ \exp(\bar{\psi} A \psi) = \int d\bar{\psi} d\psi \ \exp\left(\sum_{i,j}^{n} \bar{\psi}_i A_{ij} \psi_j\right).$$

Proof. By Theorem 2.1 and Theorem 2.2, we obtain the following product expression

$$\int d\bar{\psi} d\psi \ \exp\left(\sum_{i,j}^{n} \bar{\psi}_i A_{ij} \psi_j\right) = \int d\bar{\psi} d\psi \prod_{i,j}^{n} (1 + \bar{\psi}_i A_{ij} \psi_j)$$

which evaluates all ordered pairs (i, j) such that $i, j \in \{1, 2, ..., n\}$, n being the dimension of the square matrix A. Note that $d\bar{\psi} d\psi$ is analogous to its counterpart found in Theorem 2.3.

Consider $\prod_{i,j}^{n} (1 + \bar{\psi}_i A_{ij} \psi_j)$, a product of binomials. Note that each integrated summand in the expanded polynomial (i.e. of the form $a\chi_1 \chi_2 \dots \chi_n$) is the product of $2n$ variables, or n binomial factors in the form $\bar{\psi}_i A_{ij} \psi_j$. The coefficient of the collective sum of these summands of degree $2n$ equals the integral value. Since $\psi_j^2 = 0$, we only need to take into account the expanded terms where each variable is multiplied only once. For these terms, both i and j must take the value of every integer in the set $\{1, 2, \dots, n\}$ for a summand with a nonnegative constant to be made. This is the same situation as trying to choose n entries of A (in the form A_{ij}) such that no entries share the same row or column. Furthermore, the constant coefficient in the former and the product of the entries in the latter are equal: a product of n A_{ij} terms. Taking the summation of all products of n A_{ij} terms satisfying the above conditions yields the desired result. ■

Next, we introduce a corollary for a more intuitive graphical analysis of MGA.

Corollary 3.1.1. *For an $n \times n$ matrix A,*

$$perm(A) = \int d\bar{\psi} d\psi \prod_{i,j}^{n} (1 + (\bar{\psi}_i - \bar{\psi}_j) A_{ij} \psi_j) \prod_{i,j}^{n} (1 + \bar{\psi}_j B_j \psi_j),$$

where B_j is the sum of the terms in column j of A.

Proof. Note that

$$\sum_{i,j}^{n} \bar{\psi}_i A_{ij} \psi_j = \sum_{j=1}^{n} \bar{\psi}_j \left(\sum_{i=1}^{n} A_{ij} \right) \psi_j + \sum_{i,j}^{n} (\bar{\psi}_i - \bar{\psi}_j) A_{ij} \psi_j = \sum_{j=1}^{n} B_j \bar{\psi}_j \psi_j + \sum_{i,j}^{n} (\bar{\psi}_i - \bar{\psi}_j) A_{ij} \psi_j.$$

Thus,

$$\begin{aligned} \text{perm}(A) = \int d\bar{\psi} d\psi \ \exp\left(\sum_{i,j}^{n} \bar{\psi}_i A_{ij} \psi_j \right) &= \int d\bar{\psi} d\psi \ \exp\left(\sum_{j=1}^{n} B_j \bar{\psi}_j \psi_j + \sum_{i,j}^{n} (\bar{\psi}_i - \bar{\psi}_j) A_{ij} \psi_j \right). \\ &= \int d\bar{\psi} d\psi \ \exp\left(\sum_{j=1}^{n} B_j \bar{\psi}_j \psi_j \right) \exp\left(\sum_{i,j}^{n} (\bar{\psi}_i - \bar{\psi}_j) A_{ij} \psi_j \right) \\ &= \int d\bar{\psi} d\psi \prod_{j=1}^{n} \left(1 + B_j \bar{\psi}_j \psi_j\right) \prod_{i,j}^{n} \left(1 + (\bar{\psi}_i - \bar{\psi}_j) A_{ij} \psi_j\right). \end{aligned}$$

Rearranging yields the desired results.

■

Theorem 3.2 (Bapat)**.** *Let G be a graph with n vertices. Let S denote the set of subgraphs H of G such that H has no odd cycles and in each connected component of H, the number of vertices is equal to the number of edges. For each such subgraph H, let $c(H)$ and $c_0(H)$ denote the number of cycles in H and the number of cycles with length 2, respectively. Then for the Laplacian L of G,*

$$perm(L) = \sum_{H \in S} 2^{2c(H) - c_0(H)}.$$

Proof. By Corollary 3.1.1,

$$\text{perm}(L) = \int d\bar{\psi} d\psi \prod_{j=1}^{n} \left(1 + B_j \bar{\psi}_j \psi_j\right) \prod_{i,j}^{n} \left(1 + (\bar{\psi}_i - \bar{\psi}_j) L_{ij} \psi_j\right) = \int d\bar{\psi} d\psi \prod_{i,j}^{n} \left(1 + (\bar{\psi}_i - \bar{\psi}_j) L_{ij} \psi_j\right)$$

$$= \int d\bar{\psi} d\psi \prod_{i,j|(v_i,v_j)\in E}^{n} \left(1 + (\bar{\psi}_j - \bar{\psi}_i)\psi_j\right).$$

For each pair of Grassmann variables χ_i and χ_j, we can interpret ψ_j as vertex v_j and $\left(\bar{\psi}_j - \bar{\psi}_i\right)$ as a directed edge from v_i to v_j.

Let H be a subgraph enumerated by the permanent. The integral evaluates the coefficient of $\psi_1\psi_2\ldots\psi_n\bar{\psi}_1\bar{\psi}_2\ldots\bar{\psi}_n$. These terms are produced by a combination of n factors in the form $(\bar{\psi}_j - \bar{\psi}_i)\psi_j$ from the binomial expression. Since ψ_j must be distinct in each of the n terms, every subgraph enumerated by the integral must be such that for each i in which $1 \leq i \leq n$, $\text{indeg}(v_i) = 1$. Thus, in each component of a subgraph, every vertex has a corresponding edge (the single directed edge that contributes to its indegree), and thus the number of vertices is equal to the number of edges in each component of H.

Now we will prove H cannot have an odd cycle via the following lemma.

Lemma 3.1. For modified Grassmann variables $\psi_1, \psi_2, \ldots, \psi_k$,

$$(\psi_2 - \psi_1)(\psi_3 - \psi_2)\ldots(\psi_k - \psi_{k-1})(\psi_1 - \psi_k) = \begin{cases} 0 & \textit{if } k \textit{ is odd} \\ 2\psi_1\psi_2\ldots\psi_k & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. $\psi_i^2 = 0$ for all $1 \leq i \leq n$, so the expression only returns $\psi_1\psi_2\ldots\psi_n$ terms. Thus,

$$(\psi_2 - \psi_1)(\psi_3 - \psi_2)\ldots(\psi_n - \psi_{n-1})(\psi_1 - \psi_n) = \psi_1\psi_2\ldots\psi_n + (-1)^n\psi_1\psi_2\ldots\psi_n.$$

We can easily obtain the desired result by considering the parity of n.

Thus, for each cycle of length 2 in H, H is enumerated by a factor of 2 by Lemma 3.1. However, for each cycle with length greater than 2, H is enumerated instead by a factor of 4 because we consider two binomial products, $(\psi_2 - \psi_1)(\psi_3 - \psi_2)\ldots(\psi_k - \psi_{k-1})(\psi_1 - \psi_k)$

and $(\psi_1 - \psi_2)(\psi_2 - \psi_3)\dots(\psi_{k-1} - \psi_k)(\psi_k - \psi_1)$, with each product enumerating H twice. Therefore, H is enumerated by a factor of $2^{c_0(H)}4^{c(H)-c_0(H)}$, or $2^{2c(H)-c_0(H)}$, as desired. ■

The subgraphs enumerated by Bapat's theorem are by definition also bipartite graphs, since no odd cycles exist. Therefore, the theorem and our more intuitive method for its proof have applications in many fields that require enumeration of bipartite graphs, such as query writing [2] and analysis of data clusters [6].

Theorem 3.3. *Let G be a graph. Let I and J denote subsets of $\{1, 2, \dots, n\}$ of size k. Let H be a subgraph of G satisfying the following properties:*

- *There are k tree-components (components that are trees).*
- *For each tree-component, there exists a unique $i \in I$ and $j \in J$ such that v_i and v_j are connected by a path within that tree-component.*
- *Non-tree components have an equal number of vertices as edges, without odd cycles.*

For each subgraph H, let $\sigma(H)$ denote the sum of the lengths of the unique paths connecting each v_i and v_j in each tree-component. Let $c(H)$ and $c_0(H)$ denote the number of cycles in H and the number of cycles with length 2, respectively. If L is the Laplacian matrix of G, then for the $n-k$ by $n-k$ submatrix L_{IJ},

$$perm(L_{IJ}) = \sum_{H \in S} (-1)^{\sigma(h)} 2^{2c(H)-c_0(H)}.$$

Proof. Let G be a graph, and let L be its Laplacian matrix. First, an essential lemma:

Lemma 3.2. Using the notation $\psi_S = \prod_{s \in S} \psi_s$, where S is an integer set,

$$\text{perm}(L_{IJ}) = \int d\bar{\psi} d\psi \; \psi_I \bar{\psi}_J \prod_{i,j}^{n} (1 + (\bar{\psi}_i - \bar{\psi}_j) L_{ij} \psi_j) = \int d\bar{\psi} d\psi \; \psi_I \bar{\psi}_J \prod_{i,j}^{n} (1 + \bar{\psi}_i L_{ij} \psi_j).$$

Proof. By Corollary 3.1.1, the integral expressions in the lemma statement are equivalent. Consider the expression on the far right. The coefficient $\psi_I\bar{\psi}_J$ is comprised of $2k$ variables. Thus, the integral returns the coefficient of the summand with degree $2(n-k)$ from the product, which is exclusively composed of all ψ_c such that $c \notin I$ and all $\bar{\psi}_d$ such that $d \notin J$. Combinatorially, the integral of the product expression is equivalent to the permanent expression of a $(n-k)\times(n-k)$ matrix with rows I^C and columns J^C (where the C denotes the complement of a set). But this matrix is equal to L_{IJ}, so the coefficient of the summand is $\text{perm}(L_{IJ})$, proving the lemma.

Recall that

$$perm(L_{IJ}) = \int d\bar{\psi}d\psi\ \psi_I\bar{\psi}_J \prod_{(v_i,v_j)\in E}(1+(-1)(\bar{\psi}_i-\bar{\psi}_j)\psi_j),$$

where E is the edge set of G. Recall the previous graphical interpretation for $(\bar{\psi}_i-\bar{\psi}_j)\psi_j$, which represents a directed edge (v_i, v_j). The product expression represents a collection of edges, i.e. a graph. Note that each summand in the binomial expansion contributing to the integral value is a product of $n-k$ terms in the form $(-1)(\bar{\psi}_i-\bar{\psi}_j)\psi_j$. We now show that the subgraphs described in the problem statement correspond directly to the coefficient of $\psi_{IC}\bar{\psi}_{JC}$ and that all other graphs are not enumerated in the integration.

Let H be a subgraph of G with the properties given in the theorem statement. Let $I = \{i_1, \dots i_k\}$ and $J = \{j_1, \dots j_k\}$. Without loss of generality, let v_{i_a} and v_{j_a} be in the component of T_a, the a^{th} tree-component. Since trees are counted in a uniform process, we will only count the first tree, T_1, and generalize to the others.

T_1 has vertex set $v_{i_1}, v_{j_1}, v_{a_1}, \dots, v_{a_t}$ and its collection of edges contribute the product

$$\prod_{(v_i,v_j)\in E_{T_1}}^{t+1}(-1)(1+(\bar{\psi}_i-\bar{\psi}_j)\psi_j),$$

for which we will show that the coefficient of the term with degree $2t+4$ equals either 1

or -1, depending on the length of the unique path (by definition of spanning tree) from v_i to v_j. For each $\bar{\psi}$ term, j must take the value of each of the vertex indices except i_1. This implies that every other vertex has indegree equal to 1.

Now we consider the product chain of $(\bar{\psi}_i - \bar{\psi}_j)$ terms. For the binomial terms, adjacent edges will share one term in the binomial due to their common vertex. Thus, we note that the simplification of this chain into a nonzero product always involves taking every first term or every second term of each binomial $(\bar{\psi}_i - \bar{\psi}_j)$. Note that either the $\bar{\psi}_{i_1}$ or the $\bar{\psi}$'s for each of the nodes with outdegree 0 will not be present in the thus incomplete product.

To resolve these shortcomings in the product, we modify our product such that a $\psi_{i_1}\bar{\psi}_{j_1}$ is multiplied outside of the product. This achieves two purposes: the ψ_{i_1} term in the integral is now accounted for, and the simplification of the binomial chain is no longer 0 with the lone factor of $\bar{\psi}_{j_1}$.

The product is affected in that instead of evaluating to the product of every first term and the product of every second term, $\bar{\psi}_{j_1}$ means all other instances of that variable are not included in the product, effectively deciding which variable in the binomial is chosen depending on where the edge is in relation to v_{j_1}. All directed edges of paths from v_{j_1} to the nodes with indegree 0 have every second binomial term multiplied, and all remaining edges have every first binomial term multiplied (this follows directly from the placement of the $\bar{\psi}_{j_1}$ for these two distinct types of edges), yielding a term of degree $2t + 4$.

Now we take into account the sign of the coefficient. Note that for T_1, the sign is equal to $(-1)^{t+2}(-1)^{(t+2)-l} = (-1)^l$, where l is the length of the path from v_{i_1} to v_{j_1}. Thus, the sign of the enumeration is negative when l is odd and positive when l is even for T_1.

It becomes clear that any component that does not fit the criteria of the theorem statement will necessarily not have a term of degree $2t+4$; in fact, such components would have a coefficient of 0, which, when enumerated over all components in H, would lead to the count of the whole subgraph being 0.

We can enumerate every subgraph via this method for each tree-component, and the

remaining components with equal numbers of edges and vertices are enumerated by an argument analogous to that of Theorem 3.2 (Bapat's Theorem). Thus, in any subgraph H, H is enumerated positively or negatively depending on the sum of l values in each component, and even cycles (including cycles of length 2), which can be present in non-tree components, contribute additional enumerative factors. Taking the summation for the enumeration of all subgraphs, we have $\int d\bar{\psi}d\psi\ \psi_I\bar{\psi}_J \prod_{i,j}^{n}(1+(\bar{\psi}_i-\bar{\psi}_j)L_{ij}\psi_j) = \sum_{H\in S}(-1)^{\sigma(h)}2^{2c(H)-c_0(H)}$. ■

Theorem 3.3 enables counting all subgraph solutions for situations where each object in a set of objects must be paired with an object from a second object set and connected in the most efficient manner. Evaluating permanent minors can generate non-intersecting paths (i.e. no shared vertex in any two paths) between pairs of vertices, offering a powerful tool to optimize networks and circuits by connecting specific pairs of objects when necessary.

3.2 Positive Laplacian Matrix

Laplacian matrices (and their minors) have proven useful in enumerating certain types of subgraphs, enabling an accurate evaluation of the connectivity of a graph. We proceed to modify the Laplacian matrix definition in order to derive additional enumerations, particularly those that include subgraphs with odd cycles (not necessarily bipartite).

Definition 3.2. For a graph with n vertices, the *positive Laplacian matrix* of a graph L^+ is an $n \times n$ matrix where

$$L^+_{ij} = \begin{cases} deg(\mathrm{v}_i) & \text{if } i = j \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the sum of terms in row i or column i is equal to $2deg(v_i)$.

Theorem 3.4. *Let G be a graph with n vertices. Let S denote the set of subgraphs H of G such that in each connected component of H, the number of vertices and edges is equal. For*

each such subgraph H, let $c(H)$ and $c_0(H)$ denote the number of cycles in H and the number of cycles with length 2, respectively. Then for the positive Laplacian L^+ of G,

$$perm(L^+) = \sum_{H \in S} 2^{2c(H)-c_0(H)}.$$

Proof. Recall that

$$\text{perm}(L) = \int d\bar{\psi}d\psi \ \exp\left(\sum_{i,j}^{n} \bar{\psi}_i L_{ij}\psi_j\right) = \int d\bar{\psi}d\psi \ \exp\left(\sum_{j=1}^{n} B_j\bar{\psi}_j\psi_j + \sum_{i,j}^{n}(\bar{\psi}_i - \bar{\psi}_j)L_{ij}\psi_j\right)$$

$$= \int d\bar{\psi}d\psi \ \exp\left(\sum_{j=1}^{n} 2deg(v_j)\bar{\psi}_j\psi_j + \sum_{i,j}^{n}(\bar{\psi}_i - \bar{\psi}_j)L_{ij}\psi_j\right)$$

$$= \int d\bar{\psi}d\psi \ \exp\left(\sum_{j=1}^{n} \bar{\psi}_j\left(2\sum_{i=1}^{n} L_{ij}\right)\psi_j + \sum_{i,j}^{n}(\bar{\psi}_i - \bar{\psi}_j)L_{ij}\psi_j\right)$$

$$= \int d\bar{\psi}d\psi \ \exp\left(\sum_{i,j}^{n}(\bar{\psi}_i + \bar{\psi}_j)L_{ij}\psi_j\right)$$

$$= \int d\bar{\psi}d\psi \ \prod_{i,j}^{n}\left(1 + \left(\bar{\psi}_i + \bar{\psi}_j\right)L_{ij}\psi_j\right).$$

Here we interpret ψ_j as vertex v_j and $\left(\bar{\psi}_j + \bar{\psi}_i\right)$ as directed edge (v_i, v_j).

With an argument analogous to that in Theorem 3.2, we can conclude that each H has an equal number of vertices and edges in each of its components.

Lemma 3.3. For modified Grassmann variables $\psi_1, \psi_2, \ldots, \psi_k$,

$$(\psi_2 + \psi_1)(\psi_3 + \psi_2)\ldots(\psi_k + \psi_{k-1})(\psi_1 + \psi_k) = 2\psi_1\psi_2\ldots\psi_k.$$

Proof. Since $\psi_i^2 = 0$ for all $1 \leq i \leq n$, the binomial expression simplifies to the desired result.

Thus, for each cycle of length 2 in H, H is enumerated by a factor of 2 by Lemma 3.3. For each cycle with length greater than 2, H is enumerated by a factor of 4 because we consider two distinct binomial products in the product expansion, $(\psi_2+\psi_1)(\psi_3+\psi_2)\ldots(\psi_k+$

$\psi_{k-1})(\psi_1+\psi_k)$ and $(\psi_1+\psi_2)(\psi_2+\psi_3)\ldots(\psi_{k-1}+\psi_k)(\psi_k+\psi_1)$, with each product enumerating H twice. Therefore, H is enumerated by a factor of $2^{c_0(H)}4^{c(H)-c_0(H)}$, or $2^{2c(H)-c_0(H)}$, as desired. ■

4 Applications

While there are countless real-world situations that can be represented by even-cycled graphs such as those we have focused on in our research, there has been limited research into the importance and usage of the permanent of the Laplacian matrix representing the graph.

Theorem 3.3 offers a new way to count subgraphs with the connection of any number of vertex pairs as a requirement. While algorithms do exist to detect the shortest path between any two vertices, our theorem is able to consider several vertex pairs connected efficiently in disjoint trees. These subgraph criteria can potentially align certain network requirements to optimize data transfer efficiency [11].

The enumerations of cycles and paths achieved by Theorem 3.3 and Theorem 3.4 also have strong applications in bioinformatics. Klamt and von Kamp recognize the necessity of path and cycle enumerations to evaluate biological causal relationships in their research [10]. They work with interaction graphs, which are signed, directed graphs representing one-on-one connectivity. They state the requirement of path-finding and cycle-counting in specific biological networks such as cell signaling, protein-protein interactions, complex feedback loops, and signal transduction. Through our enumerations, we have presented an alternative way of computing such counts through matrix calculations instead of their preferred backtracking algorithms. In fact, since the number of paths starting from a particular vertex can increase exponentially, their breadth-first algorithms are impractical in extensive enumeration; however, computation of the Laplacian permanent still holds to be a feasible enumerator. Though Klamt and von Kamp mention a lack of research done on cyclical enumerations, our Theorem 3.4 applies itself perfectly, being able to count collections of cycles

of both even and odd length.

In query rewriting, weighted bipartite graphs, where one vertex set represents queries and the other represents ads, are used to represent click graphs [2]. These click graphs are crucial for leading search engines such as Google and Yahoo to produce the most relevant search results. By studying connections between what users type and what their click actions show they are looking for, search engines can predict search preferences for users. Our research extends this use of bipartite click graphs to count the number of possible query and ad connections to further the efficiency and practicality of query rewriting in search engines. Our findings in the combinatorial interpretation of the permanent of click graphs may be used to further refine and specify search results to give users the most optimal return.

5 Future Work

We have demonstrated through our results that Grassmann-Berezin and Modified Grassmann-Berezin Calculus are far more suitable for combinatorial arguments than what most mathematicians have expected them to be. Our results may also shed light on the bounding of the permanent of any matrix, a problem that has garnered the attention of mathematicians for decades [4], [8]. Our relation between the Laplacian permanent with cycles also suggests the possibility of reverse-engineering an algorithm for a faster computation of the permanent. Such an extension of our work would be monumental in the field of computer science. There is much more to the Laplacian matrix than its determinant and permanent; for example, the combinatorial interpretation of the square root of the graph Laplacian is an equally elusive result, although a systematic application of the Binomial Theorem on Grassmann variables may be likely to produce an algebraic expression.

Overall, the capacity of this unusual algebra (which had rarely been used outside of its purpose in physics) to excavate these previously unseen mathematical connections serves as a beacon of inspiration that will guide further inquiries of the field.

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