Binomial Coefficients

Binomioid Coefficients

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Pascal's Triangle

Binomial coefficients $\binom{n}{k}$ are entries in Pascal's Triangle.

	0	1	2	3	4	5	6	7	8	
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Index $n \geq 0$ vertically on the left, and index $k \geq 0$ across the top.

First called *Pascal's Triangle* in 1708, after French scholar Blaise Pascal (1623 - 1662). But it arose earlier.

Italians call it *Tartaglia's triangle.* After Niccolò Tartaglia writing in 1556.

Levi ben Gerson used factorial formula around 1300.

Called the *Khayyam triangle* in Iran. Persian Omar Khayyam $\sim$ 1100.

Called *Yang Hui's triangle* in China. Yang Hui 杨辉 (1238 – 1298) studied that arithmetic triangle, following Jia Xian 贾宪 $\sim$ 1050.

Mahāvīra wrote the factorial formula for $\binom{n}{k}$, around 850.

Around 100 BCE, Indian scholar Pingala studied meters in Sanskrit prosody, and wrote about that arithmetic triangle.

As well as: zero, binary numerals, binomial theorem, and Fibonacci numbers.

Three standard ways to define those entries $\binom{n}{k}$.

- Count how many k-sets are in an n-set.
- Additive Recursion: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.
- Factorials: $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ \cdots\ (n-k+1)}{k(k-1)\ \ \cdots\ \ 1}$.

With another sequence in place of $(n) = (1, 2, 3, 4 \ldots)$, that factorial formula provides an analogue of binomial coefficients.

Let $(c) = (c_1, c_2, c_3, \ldots)$ be a sequence of positive integers. Define the c-**factorial** to be: $\langle n\rangle!_c = c_n c_{n-1} \cdots c_2 c_1$.

For $(c) = (c_1, c_2, \dots)$, define the c-**nomial coefficients**

$$\begin{bmatrix} n \\ k \end{bmatrix}_c = \frac{\langle n \rangle!_c}{\langle k \rangle!_c \cdot \langle n-k \rangle!_c} = \frac{c_n c_{n-1} \cdots c_{n-k+1}}{c_k \, c_{k-1} \quad \cdots \quad c_1}.$$

These numbers form the **Pascaloid Triangle** for c.

Note: $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$, and rows are symmetric: $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$.
If $k > n$, define $\begin{bmatrix} n \\ k \end{bmatrix} = 0$.

Definition

Sequence (c) is **binomioid** if all $\begin{bmatrix} n \\ k \end{bmatrix}_c$ are integers.

Binomioid examples.

- Constant sequence $(3, 3, 3, 3, \dots)$ is binomioid.

Check: $\begin{bmatrix} n \\ k \end{bmatrix}_C = \frac{c_n c_{n-1} \cdots c_{n-k+1}}{c_k\, c_{k-1} \cdots c_1} = \frac{3^k}{3^k} = 1.$

- $\{$ binomioid sequences $\}$ is closed under multiplication.

That is: If $(a) = (a_1, a_2, \dots)$ and $(b) = (b_1, b_2, \dots)$ are binomioid, then the product sequence $(ab) = (a_1 b_1, a_2 b_2, \dots)$ is also binomioid.

Corollary. Sequence $(n^2) = (1, 4, 9, 16, \dots)$ is binomioid.

Choose your favorite sequence, construct its Pascaloid Triangle, and see whether it is binomioid.

Before going to more examples, let's review why
$(n) = (1, 2, 3, \dots)$ is binomioid.
The usual proof uses counting or recursion.
We outline a purely **factorial proof**.

For a prime p, what power of p occurs in $n!$?

Lemma. $\boxed{(\text{Exponent of } p \text{ in } n!) = \sum_{k \geq 1} \lfloor n/p^k \rfloor.}$

Proof. In $1, 2, \dots, n$, count $\lfloor n/p \rfloor$ terms with p as a factor.
But some terms involve more than one factor of p.
Count $\lfloor n/p^2 \rfloor$ with p^2 as a factor. Etc.
So: $(\text{Exponent of } p \text{ in } n!) = \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$. QED

We outline a purely **factorial proof** that the sequence $(n) = (1, 2, 3, \dots)$ is binomioid.

To prove: Every $\binom{a+b}{a}$ is an integer.

Suffices to check, for every prime p:

$$\text{Exponent of } p \text{ in } \frac{(a+b)!}{a!\,b!} \text{ is } \geq 0.$$

This follows from the Lemma, if we verify the following exercise. QED

Exercise. If α, β are real numbers then:
$$\lfloor \alpha + \beta \rfloor \geq \lfloor \alpha \rfloor + \lfloor \beta \rfloor.$$

Easy Examples.

• How about $(c) = (2^n) = (2, 4, 8, 16, \dots)$?

Is every $\begin{bmatrix} n \\ k \end{bmatrix}_C = \frac{2^n}{2^k} \frac{2^{n-1}}{2^{k-1}} \cdots \frac{2^{n-k+1}}{2^1}$ an integer?.

In general, notice that: $\begin{bmatrix} n \\ k \end{bmatrix}_C = \frac{c_n}{c_k} \frac{c_{n-1}}{c_{k-1}} \cdots \frac{c_{n-k+1}}{c_1}$

If $c_r \mid c_s$ whenever $r \leq s$, then this product is an integer.

Definition. $c = (c_n)$ is a *divisor-chain* if $c_n \mid c_{n+1}$ for every n.

We have proved: $\boxed{\text{Every divisor-chain is binomioid.}}$

More challenging examples.

• Is $(2^n - 1) = (1, 3, 7, 15, \dots)$ binomioid?

For example, $\begin{bmatrix} 6 \\ 3 \end{bmatrix}_C = \left[\frac{2^6-1}{2^3-1}\right]\left[\frac{2^5-1}{2^2-1}\right]\left[\frac{2^4-1}{2^1-1}\right] = \frac{63}{7}\frac{31}{3}\frac{15}{1} = 1395.$

Prove: Every $\begin{bmatrix} n \\ 3 \end{bmatrix}_C = \left[\frac{2^n-1}{7}\right]\left[\frac{2^{n-1}-1}{3}\right]\left[\frac{2^{n-2}-1}{1}\right]$ is an integer.

Idea goes back 1808 with Gauss's "q-nomial coefficients":

Theorem. $\boxed{(q^n - 1) \text{ is binomioid, for any integer } q > 1.}$

Perhaps this is not surprising because we know:

$$m \mid n \implies (q^m - 1) \mid (q^n - 1).$$

(How to prove that?)

Exercise. If sequence (a) satisfies that divisibility property ($m \mid n \Rightarrow a_m \mid a_n$), must (a) be binomioid?

Gauss studied factorizations of $x^n - 1$.

$$\begin{aligned}
x^1 - 1 &= (x-1) \\
x^2 - 1 &= (x-1)(x+1) \\
x^3 - 1 &= (x-1)(x^2+x+1) \\
x^4 - 1 &= (x-1)(x+1)(x^2+1) \\
x^5 - 1 &= (x-1)(x^4+x^3+x^2+x+1) \\
x^6 - 1 &= (x-1)(x+1)(x^2+x+1)(x^2-x+1)
\end{aligned}$$

Those factors are the **cyclotomic polynomials** $\Phi_n(x)$. Then

$$\begin{aligned}
\Phi_1(x) &= x - 1 & \Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 & \Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 & \Phi_6(x) &= x^2 - x + 1
\end{aligned}$$

Generally, $\Phi_n(x) \in \mathbb{Z}[x]$ is monic polynomial of degree $\varphi(n)$ and

$$\boxed{x^n - 1 = \prod_{d|n} \Phi_d(x),}$$

Divisor Products.

Definition. Sequence $(a) = (a_1, a_2, \ldots)$ is a *divisor-product* if there is an integer sequence (b) such that

$$a_n = \prod_{d|n} b_d, \quad \text{for every } n. \qquad (*)$$

For instance,

$$\begin{array}{lll} a_1 = b_1 & a_2 = b_1 b_2 & a_3 = b_1 b_3 \\ a_4 = b_1 b_2 b_4 & a_5 = b_1 b_5 & a_6 = b_1 b_2 b_3 b_6 \end{array}$$

If every $a_n \neq 0$, solve successfully for b_n. (Compare Möbius Inversion.)
There always exist $b_n \in \mathbb{Q}$ satisfying equations (*).
Divisor-products have all $b_n \in \mathbb{Z}$.

Divisor-Product Theorem.

$$\boxed{\text{Every divisor-product sequence is binomioid.}}$$

For integer $q > 1$, observe that:
the sequence $q^n - 1$ is a divisor-product.

Because $q^n - 1 = \prod_{d|n} \Phi_d(q)$ and each $\Phi_d(q) \in \mathbb{Z}^+$.

Therefore: (Divisor-Product Theorem) $\Rightarrow$ (Gauss's Theorem).

For example, $2^n - 1 = \prod_{d|n} g_d$,
where $(g_n) = \big(\Phi_n(2)\big) = (1, 3, 7, 5, 31, 3, 127, 17, 73, 11, \dots)$.

Exercise. Do these ideas generalize to show that
$(3^n - 2^n)$ is a divisor-product?

Exercise. Which of the following are divisor-products?

- Constant sequence $(c, c, c, \dots)$.
- $(c^n) = (c, c^2, c^3, \dots)$.
- $(n) = (1, 2, 3, \dots)$.
- Fibonacci $(1, 1, 2, 3, 5, 8, 13, \dots)$.
- Euler function $(\varphi(n)) = (1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, \dots)$.
- Any divisor-chain.

Hint. One answer for this exericise is "No."

Is the set $\{$ divisor-products $\}$ closed under multiplication?

PROOF of the Divisor-Product Theorem.

Given $a_n = \prod_{d|n} b_d$. View the symbols $b_1, b_2, \ldots$ as indeterminates.

Every $\left[{r+s \atop r} \right]_a$ is a monomial fraction in $b_1, b_2, \ldots$

To prove: Denominators cancel. That is:

For every k, exponent of b_k in that fraction is ≥ 0.

- First: b_k is a factor of $a_m \iff k \mid m$. (And b_k^2 is not a factor.)
- Second: $\langle n \rangle! = a_n a_{n-1} \cdots a_2 a_1$ so that:
$$\big(\text{Exponent of } b_k \text{ in } \langle n \rangle!\big) = \lfloor n/k \rfloor.$$
- Finally: Exponent of b_k in $\left[{r+s \atop r} \right] = \frac{\langle r+s \rangle!}{\langle r \rangle! \langle s \rangle!}$ equals

  $\lfloor \frac{r+s}{k} \rfloor - \lfloor \frac{r}{k} \rfloor - \lfloor \frac{s}{k} \rfloor$.

  That is always ≥ 0, by earlier Exercise. QED

NEW EXAMPLES. $(n+1) = (2, 3, 4, \ldots)$ is NOT binomioid.

Instead let's try the product sequence:

- $(n(n+1)) = (2, 6, 12, 20, 30, 42, \ldots)$.

It's Pascaloid triangle looks like

	0	1	2	3	4	5	6	7	
0	1								
1	1	1							
2	1	3	1						
3	1	6	6	1					
4	1	10	20	10	1				
5	1	15	50	50	15	1			
6	1	21	105	175	105	21	1		
7	1	28	196	490	490	196	28	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Exercise. Prove all those entries are integers.

Compute entry $\left[\begin{matrix} n \\ k \end{matrix}\right] = \frac{1}{k+1}\binom{n+1}{k}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}$.

If true, then the "triangular number" sequence $\left(\frac{n(n+1)}{2}\right) = (1, 3, 6, 10, 15, 21, \dots)$ is binomioid.

That sequence is Column #2 in the classic Pascal Triangle:

	0	1	2	3	4	5	6	7	8	
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then Columns 0, 1, 2 are binomioid.

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	0	1	2	3	4	5	6	7	8	
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then Columns 0, 1, 2 are binomioid.

How about Column 3 = (1, 4, 10, 20, . . .) ?

Here's the Pascaloid triangle for Pascal Column #3:

$$\left(\binom{n+2}{3}\right) = \left(\frac{n(n+1)(n+2)}{6}\right) = (1, 4, 10, 20, 35, 56, \dots)$$

	0	1	2	3	4	5	6	7	
0	1								
1	1	1							
2	1	4	1						
3	1	10	10	1					
4	1	20	50	20	1				
5	1	35	175	175	35	1			
6	1	56	490	980	980	56	1		
7	1	84	1176	4116	4116	1176	84	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Seems to be binomioid, and we wonder:

$\boxed{\text{What about the other Pascal columns?}}$

Theorem

Every column of Pascal's Triangle is binomioid.

Our usual proof Idea.

Each Pascal column (c) is a sequence of binomial coefficients. Then each entry of the Pascaloid Triangle for (c) is a fraction whose numerator and denominator are products f various binomial coefficients.

Compute the exponent of prime p occurring in $\begin{bmatrix} n \\ k \end{bmatrix}_c$. That calculation is complicated because the terms themselves are messy. But it works. (I think.)

What about Pascal rows?

Row 3 is $(1, 3, 3, 1)$ but that's really $(1, 3, 3, 1, 0, 0, \ldots)$.

But there are zeros in denominators for some $\begin{bmatrix} n \\ k \end{bmatrix}$!

(More zeros in numerator than in denominator $\Rightarrow$ set quotient = 0.)

Pascaloid triangles for Pascal rows (suppressing zero entries)

For $(1, 3, 3, 1)$:

	0	1	2	3	4
0	1				
1	1	1			
2	1	3	1		
3	1	3	3	1	
4	1	1	1	1	1

For $(1, 4, 6, 4, 1)$:

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	4	1			
3	1	6	6	1		
4	1	4	6	4	1	
5	1	1	1	1	1	1

Those arrays indicate some rotational symmetry.

(Rewrite triangles to be equilateral to get better optics).

Exercise. Why does that always happen?

Here are Pascaloid triangles for the first few Pascal rows.

1			
1	1		
1	2	1	
1	1	1	1

1				
1	1			
1	3	1		
1	3	3	1	
1	1	1	1	1

1					
1	1				
1	4	1			
1	6	6	1		
1	4	6	4	1	
1	1	1	1	1	1

1						
1	1					
1	5	1				
1	10	10	1			
1	10	20	10	1		
1	5	10	10	5	1	
1	1	1	1	1	1	1

1							
1	1						
1	6	1					
1	15	15	1				
1	20	50	20	1			
1	15	50	50	15	1		
1	6	15	20	15	6	1	
1	1	1	1	1	1	1	1

1								
1	1							
1	7	1						
1	21	21	1					
1	35	105	35	1				
1	35	175	175	35	1			
1	21	105	175	105	21	1		
1	7	21	35	35	21	7	1	
1	1	1	1	1	1	1	1	1

Have we seen those numbers before?

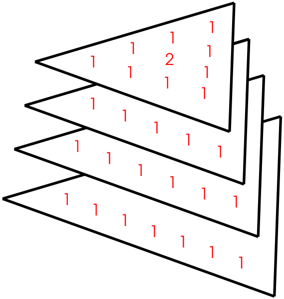
Here again is the Pascaloid triangle for Pascal's second column $\binom{n-1}{2} = (1, 3, 6, 10, 15, 21, \dots)$.

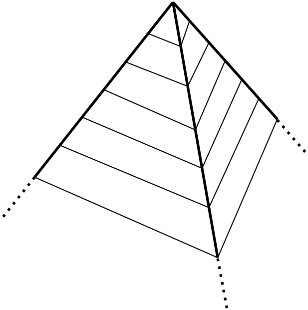
	0	1	2	3	4	5	6	7	
0	1								
1	1	1							
2	1	3	1						
3	1	6	6	1					
4	1	10	20	10	1				
5	1	15	50	50	15	1			
6	1	21	105	175	105	21	1		
7	1	28	196	490	490	196	28	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

- Each row here is Column #2 in one of the triangles above.

Note. Columns of the classical Pascal's Triangle are binomioid.
But that behavior is unusual:
In the triangle above, column 2 $(1, 6, 20, 50, 105, \dots)$ is NOT binomioid.

The Pascal rows produced a sequence of triangles. Stack them to obtain an infinite pyramid of numbers:

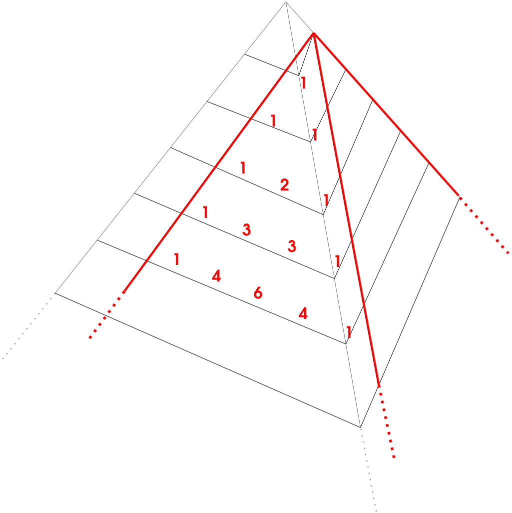




This pyramid has 3 faces. Only ones are visible on those faces, with larger numbers inside.

This number-pyramid has 3-fold rotational symmetry.

Remove one triangular face of ones, and the next visible face is the classic Pascal triangle.



What is the next triangular layer, cutting one step deeper?

Binomioid Pyramid Questions.

- What number is in position (n, k, m)?
  That is, at the $\begin{bmatrix} n \\ k \end{bmatrix}$ position on the m^{th} level.

- Each entry of this Pyramid is a positive integer equal to a fraction involving several $\binom{n}{k}$'s.

  ¿ Is there a combinatorial interpretation ?
  Does entry (n, k, m) = the size of some natural set ?

  Possibly this will lead to a different proof that all entries are integers.

General Binomioid Pyramids.

Every sequence (c) has its Pascaloid Triangle Δ_C.

Each *row* of triangle Δ_C has its own Pascaloid triangle.
Stack those to form the **Binomioid Pyramid for (c)**.

- There is 3-fold symmetry.
- Pascaloid Triangles for the *columns* of Δ_C appear as slices of that pyramid.
- Entry at position (n, k, m) is a fraction involving products of various $\begin{bmatrix} r \\ s \end{bmatrix}_C$ terms.

Definition. Sequence (c) is *binomioid at all levels* if all entries of its Binomioid Pyramid are integers.

Binomioid Pyramid Questions.

$(n) = (1, 2, 3, \dots)$ is binomioid at all levels.

- Are divisor-chains always binomioid at all levels?
- Are divisor-products always binomioid at all levels?
- Any other examples?

THE END

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