Homework assignment 5.1:

1. Solve equation 5.5. For those of you that are a little rusty on solving differential equation, please check:

   http://web.missouri.edu/~kovaleskis/ApplEMandEP/Lectures/Lecture-4.pdf

   So the differential equations can be solved after a substitution. Another more systematic method is to use the hint provided in the book. This will give the following differential equation:

$$\ddot{y} = \omega^2\left(\frac{E}{B} - \dot{y}\right)$$

Rewrite this differential equation to:

$$\ddot{y} + \omega^2\dot{y} = \omega^2\frac{E}{B}$$

Note that this is a non-homogeneous differential equation. Such a differential equation can be solved by first finding the general solution $y_h(t)$, i.e. solve the homogeneous differential equation:

$$\ddot{y} + \omega^2\dot{y} = 0$$

Rather than solving y, it is easier to solve $\dot{y}$.

As trial solution we choose:

$$\dot{y}_h(t) = Ae^{\alpha t}$$

Leading to the characteristic equation:

$$\alpha^2 Ae^{\alpha t} + \omega^2 Ae^{\alpha t} = 0 \Leftrightarrow \alpha^2 + \omega^2 = 0 \Leftrightarrow \alpha = \pm i\omega$$

The most general solution is a linear combination of those solutions, i.e.

$$\dot{y}_h(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

Now find a particular solution for:

$$\ddot{y} + \omega^2\dot{y} = \omega^2\frac{E}{B}$$

We choose something similar to the right side, i.e. a constant:

$$\dot{y}_p = \frac{E}{B}$$

The total solution is now a linear combination of the particular and homogeneous solution, i.e.:

$$\dot{y}(t) = \dot{y}_p(t) + \dot{y}_h(t) = \frac{E}{B} + A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

Taking the integral gives:

$$y(t) = \frac{E}{B}t + \frac{A_1}{i\omega}e^{i\omega t} + \frac{A_2}{-i\omega}e^{-i\omega t} + C$$

Now use the initial condition that the particle is at rest at t=0 at y(t=0)=0 (you can consider this to be a boundary condition for the time domain). Note that we have three unknowns and only two initial conditions. There is one other conditions though, both y(t) and v_y(t) need to be real quantities. So we get:

$$y(0)=0 \Leftrightarrow \frac{A_1}{i\omega}+\frac{A_2}{-i\omega}+C=0$$

$$\dot{y}(0)=0 \Leftrightarrow A_1+A_2+E/B=0 \Leftrightarrow A_1=-A_2-E/B$$

$$A_1=A_2$$

Or in other words solving of the three constants now gives:

$$A_1=A_2=\frac{E}{2B}$$

$$C=0$$

Integration gives now the solution in y(t):

$$y(t)=\frac{E}{B}t+2\frac{E}{\omega B}\sin(\omega t)$$

2. Study the derivation of the continuity equation on the bottom of page 213 and the top of page 214. Explain the expression at the bottom of page 213 and the expression at the top of page 214.

The left side of the equation at the bottom of page 213 gives us the net exit flux (in Coulomb/sec) that leaves the enclosed surface S. In the case S is a closed surface (for example a pill box, or a sphere, or a cube), than we can write the surface integral of the vector field **J**, as a volume integral of the divergency of **J**. The integration has to be done over the complete volume that is enclosed by S.

Note that the net exit flux (in Coulomb/second) out of the closed surface should be equal to the decrease of the charge in the box:

$$\iiint_V (\nabla \bullet \vec{J})d\tau' = -\frac{\partial Q_{remaining}}{\partial t}$$

So the larger the amount of charge is that leaves the box, the lower the amount of charge is that stays behind. The amount of charge that stays behind is the volume integral of the charge density across the volume of the box, i.e.:

$$Q_{remaining}=\iiint_V \rho d\tau'$$

Combining both equations and removing the integrals gives:

$$\nabla \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

This is called the continuity equation. If the charge density is constant through space, the time derivative of the charge density is zero, and the divergency of J is equal to zero. We consider this to be the case for electrostatic cases. In chapter 7 we will see when ρ is no longer constant but varies with the time. Note that this equation also implies that charge is conserved. If we would be able to make a net positively charged object out of a neutral object, this equation would not be correct and would need an extra “magic term” on the right. You and I know better that the net charge is always conserved so no extra “magic –term” is required on the right side of above given continuity equation.

Although above equation is the continuity equation for charge, there are also continuity equations for other physical quantities. If we ignore relativity we can define a continuity equation for mass or for energy. If we consider optical waves propagating in a medium that has no absorption we could define a continuity equation for ray-optics. You will learn more about these in Chemistry, Statistical Mechanics, and optics. Note that all those continuity equations are based on conservation laws. For a lot of quantities in our society, continuity equations cannot be defined as the corresponding conservation laws do not exist. For example continuity equations for information, wealth, or joy do not really exist as those quantities are not always conserved, leading to non-zero divergencies at certain points in space. In those cases, similar to electrostatics, one could define sources and sinks of the flux version of those quantities. The brilliant scientist (source) and the forgetful person (sink), the creative entrepreneur (source) and the terrorist (sink), the funniest guy at the party (source) and the downer (sink) ☺.

3. (5.5. b) A current flows down a wire of radius a. If the current is distributed in such a way that the volume current density is inversely proportional to the distance from the axis, what is J?

   (b) $J = \frac{\alpha}{s} \Rightarrow I = \int J\,da = \alpha \int \frac{1}{s} s\,ds\,d\phi = 2\pi\alpha \int ds = 2\pi\alpha a \Rightarrow \alpha = \frac{I}{2\pi a};\ J = \boxed{\frac{I}{2\pi a s}}.$

4. (5.6 b) A uniformly charge solid sphere, of radius R and total charge Q is centered at the origin and spinning at a constant angular velocity w about the z axis. Find the current density **J** at any point (r,θ,ϕ) within the sphere.

   (b) $\mathbf{v} = \omega r \sin\theta\,\hat{\boldsymbol{\phi}} \Rightarrow \boxed{\mathbf{J} = \rho\omega r \sin\theta\,\hat{\boldsymbol{\phi}},}$ where $\rho \equiv Q/(4/3)\pi R^3$.