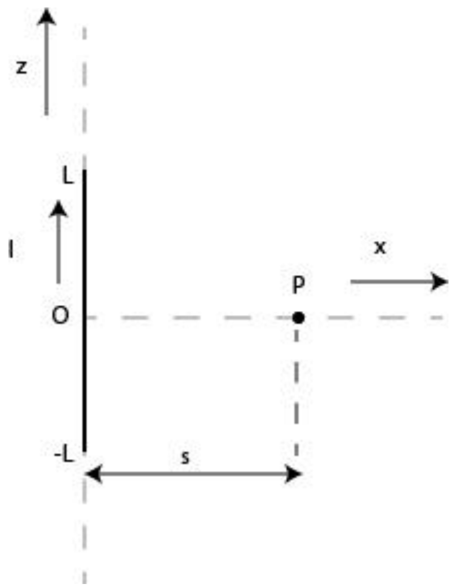
Key Homework 5.3.

1. a. A direct current I flows in a straight wire of length 2L situated along the z-axis (stretching from –L to L). Find the magnetic vector potential in a field point P that is situated in the bisecting plane (see figure below).

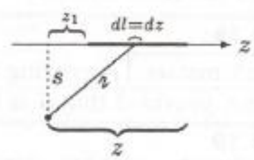


b. Use the magnetic vector potential determined in (a) to determine the magnetic field **B**.

c. Compare your answer with equation 5.35 and show that the answer is consistent with equation 5.35.

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{I\,\hat{\mathbf{z}}}{\mathscr{r}}\,dz = \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\int_{z_1}^{z_2}\frac{dz}{\sqrt{z^2+s^2}}$$

$$= \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]\Big|_{z_1}^{z_2} = \boxed{\frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{(z_2)^2+s^2}}{z_1+\sqrt{(z_1)^2+s^2}}\right]\hat{\mathbf{z}}}$$



$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A}{\partial s}\hat{\boldsymbol{\phi}} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{(z_2)^2+s^2}}\frac{s}{\sqrt{(z_2)^2+s^2}} - \frac{1}{z_1+\sqrt{(z_1)^2+s^2}}\frac{s}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2-\sqrt{(z_2)^2+s^2}}{(z_2)^2-[(z_2)^2+s^2]}\frac{1}{\sqrt{(z_2)^2+s^2}} - \frac{z_1-\sqrt{(z_1)^2+s^2}}{z_1^2-[(z_1)^2+s^2]}\frac{1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 I s}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - 1 - \frac{z_1}{\sqrt{(z_1)^2+s^2}} + 1\right]\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - \frac{z_1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}},$$

or, since $\sin\theta_1 = \dfrac{z_1}{\sqrt{(z_1)^2+s^2}}$ and $\sin\theta_2 = \dfrac{z_2}{\sqrt{(z_2)^2+s^2}}$,

$$= \boxed{\frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\,\hat{\boldsymbol{\phi}}} \text{ (as in Eq. 5.35).}$$

2. Determine the current density responsible for a magnetic vector potential described by:

$$\vec{A} = k\hat{\phi}$$

Where k is a constant.

$$A_\phi = k \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\,\hat{\mathbf{z}} = \frac{k}{s}\,\hat{\mathbf{z}};\ \mathbf{J} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \boxed{\frac{k}{\mu_0 s^2}\,\hat{\phi}.}$$

3. Assume a surface current density is flowing through the xy plane described by:

$$\vec{K} = K\hat{x}$$

In example 5.8 you derived that the magnetic field above and below the plane is given by:

$$\vec{B} = \begin{cases} \dfrac{\mu_o}{2}K\hat{y} & \text{for z} < 0 \\ -\dfrac{\mu_o}{2}K\hat{y} & \text{for z} > 0 \end{cases}$$

We noticed that this field is independent of the distance to the plane, i.e. independent of z.

What is the vector potential above the plane, and what is the vector potential below the plane.

$\mathbf{K} = K\,\hat{\mathbf{x}} \Rightarrow \mathbf{B} = \pm\frac{\mu_0 K}{2}\,\hat{\mathbf{y}}$ (plus for $z < 0$, minus for $z > 0$).

$\mathbf{A}$ is parallel to $\mathbf{K}$, and depends only on z, so $\mathbf{A} = A(z)\,\hat{\mathbf{x}}$.

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A(z) & 0 & 0 \end{vmatrix} = \frac{\partial A}{\partial z}\,\hat{\mathbf{y}} = \pm\frac{\mu_0 K}{2}\,\hat{\mathbf{y}}.$$

$\boxed{\mathbf{A} = -\frac{\mu_0 K}{2}|z|\,\hat{\mathbf{x}}}$ will do the job—or this plus any constant.