Key homework 6_3:

Problem 1:

Problem 6.16

$\oint \mathbf{H}\cdot d\mathbf{l} = I_{f_{enc}} = I$, so $\mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}$. $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \boxed{\mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\boldsymbol{\phi}}.}$ $\mathbf{M} = \chi_m\mathbf{H} = \boxed{\frac{\chi_m I}{2\pi s}\hat{\boldsymbol{\phi}}.}$

$$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\chi_m I}{2\pi s}\right)\hat{\mathbf{z}} = \boxed{0.} \quad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \boxed{\begin{cases} \frac{\chi_m I}{2\pi a}\hat{\mathbf{z}}, & \text{at } s = a; \\ -\frac{\chi_m I}{2\pi b}\hat{\mathbf{z}}, & \text{at } r = b. \end{cases}}$$

Total enclosed current, for an amperian loop between the cylinders:

$$I + \frac{\chi_m I}{2\pi a}2\pi a = (1+\chi_m)I, \quad \text{so} \quad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0(1+\chi_m)I \Rightarrow \mathbf{B} = \frac{\mu_0(1+\chi_m)I}{2\pi s}\hat{\boldsymbol{\phi}}. \checkmark$$

Problem 2:

Problem 6.17

From Eq. 6.20: $\oint \mathbf{H}\cdot d\mathbf{l} = H(2\pi s) = I_{f_{enc}} = \begin{cases} I(s^2/a^2), & (s<a); \\ I & (s>a). \end{cases}$

$$H = \left\{ \begin{array}{ll} \frac{Is}{2\pi a^2}, & (s<a) \\ \frac{I}{2\pi s}, & (s>a) \end{array} \right\}, \quad \text{so } B = \mu H = \boxed{\begin{cases} \frac{\mu_0(1+\chi_m)Is}{2\pi a^2}, & (s<a); \\ \frac{\mu_0 I}{2\pi s}, & (s>a). \end{cases}}$$

$\mathbf{J}_b = \chi_m \mathbf{J}_f$ (Eq. 6.33), and $J_f = \frac{I}{\pi a^2}$, so $\boxed{J_b = \frac{\chi_m I}{\pi a^2}}$ (same direction as I).

$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \chi_m\mathbf{H}\times\hat{\mathbf{n}} \Rightarrow \boxed{\mathbf{K}_b = \frac{\chi_m I}{2\pi a}}$ (opposite direction to I).

$I_b = J_b(\pi a^2) + K_b(2\pi a) = \chi_m I - \chi_m I = \boxed{0}$ (as it *should* be, of course).

Problem 3:

Problem 6.15

"Potentials":

$$\begin{cases} W_{\text{in}}(r,\theta) &= \sum A_l r^l P_l(\cos\theta), \quad (r<R); \\ W_{\text{out}}(r,\theta) &= \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), \quad (r>R). \end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)} & W_{\text{in}}(R,\theta) = W_{\text{out}}(R,\theta), \\ \text{(ii)} & -\frac{\partial W_{\text{out}}}{\partial r}\Big|_R + \frac{\partial W_{\text{in}}}{\partial r}\Big|_R = M^{\perp} = M\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = M\cos\theta. \end{cases}$$

(The continuity of W follows from the gradient theorem: $W(\mathbf{b}) - W(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} \nabla W \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{H}\cdot d\mathbf{l}$; if the two points are infinitesimally separated, this last integral $\to 0$.)

$$\begin{cases} \text{(i)} & \Rightarrow \; A_l R^l = \frac{B_l}{R^{l+1}} \;\Rightarrow\; B_l = R^{2l+1} A_l, \\ \text{(ii)} & \Rightarrow \; \sum (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) + \sum l A_l R^{l-2} P_l(\cos\theta) = M\cos\theta. \end{cases}$$

Combining these:

$$\sum (2l+1) R^{l-1} A_l P_l(\cos\theta) = M\cos\theta, \text{ so } A_l = 0 \ (l \neq 1), \text{ and } 3A_1 = M \Rightarrow A_1 = \frac{M}{3}.$$

Thus $W_{\text{in}}(r,\theta) = \frac{M}{3} r\cos\theta = \frac{M}{3} z$, and hence $\mathbf{H}_{\text{in}} = -\nabla W_{\text{in}} = -\frac{M}{3}\hat{\mathbf{z}} = -\frac{1}{3}\mathbf{M}$, so

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0\left(-\frac{1}{3}\mathbf{M} + \mathbf{M}\right) = \boxed{\frac{2}{3}\mu_0\mathbf{M}.} \checkmark$$

Problem 4:

Problem 6.18

By the method of Prob. 6.15:

For large r, we want $\mathbf{B}(r,\theta) \to \mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$, so $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} \to \frac{1}{\mu_0} B_0\,\hat{\mathbf{z}}$, and hence $W \to -\frac{1}{\mu_0} B_0 z = -\frac{1}{\mu_0} B_0 r\cos\theta$.

"Potentials":

$$\begin{cases} W_{\text{in}}(r,\theta) &= \sum A_l r^l P_l(\cos\theta), \quad (r<R); \\ W_{\text{out}}(r,\theta) &= -\frac{1}{\mu_0} B_0 r\cos\theta + \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), \quad (r>R). \end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)} & W_{\text{in}}(R,\theta) = W_{\text{out}}(R,\theta), \\ \text{(ii)} & -\mu_0\frac{\partial W_{\text{out}}}{\partial r}\Big|_R + \mu\frac{\partial W_{\text{in}}}{\partial r}\Big|_R = 0. \end{cases}$$

(The latter follows from Eq. 6.26.)

$$\text{(ii)} \Rightarrow \mu_0\left[\frac{1}{\mu_0} B_0\cos\theta + \sum (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta)\right] + \mu \sum l A_l R^{l-1} P_l(\cos\theta) = 0.$$

For $l \neq 1$, (i) $\Rightarrow B_l = R^{2l+1} A_l$, so $[\mu_0(l+1) + \mu l] A_l R^{l-1} = 0$, and hence $A_l = 0$.

For $l = 1$, (i) $\Rightarrow A_1 R = -\frac{1}{\mu_0} B_0 R + B_1/R^2$, and (ii) $\Rightarrow B_0 + 2\mu_0 B_1/R^3 + \mu A_1 = 0$, so $A_1 = -3B_0/(2\mu_0+\mu)$.

$$W_{\text{in}}(r,\theta) = -\frac{3B_0}{(2\mu_0+\mu)} r\cos\theta = -\frac{3B_0 z}{(2\mu_0+\mu)}. \quad \mathbf{H}_{\text{in}} = -\nabla W_{\text{in}} = \frac{3B_0}{(2\mu_0+\mu)}\hat{\mathbf{z}} = \frac{3\mathbf{B}_0}{(2\mu_0+\mu)}.$$

$$\mathbf{B} = \mu\mathbf{H} = \frac{3\mu\mathbf{B}_0}{(2\mu_0+\mu)} = \boxed{\left(\frac{1+\chi_m}{1+\chi_m/3}\right)\mathbf{B}_0.}$$