Chapter 5 Magnetostatics.

We used $\mathbf{r_s}$ for r-script.

We introduced the magnetic field to describe interaction of moving charges or interaction of currents. Once the magnetic field and electric field are known we can find the total force acting on a moving charge:

$$\vec{F} = Q\left(\vec{v} \times \vec{B} + \vec{E}\right) \qquad [1]$$

If an object consists of a lot of moving charges than of course we need to integrate over all those moving charges to find the net force:

$$\vec{F} = \int\left(\vec{v} \times \vec{B} + \vec{E}\right)dq \qquad [2]$$

We often write this equation in terms of charge density. Ignoring the electrostatic contribution, for linear, surface, and volume charge density, this equation can be converted into:

$$\begin{aligned} \vec{F} &= \int \vec{v} \times \vec{B}\lambda dl = \int \vec{I} \times \vec{B} dl = \int I\left(d\vec{l} \times \vec{B}\right) \\ \vec{F} &= \iint \vec{v} \times \vec{B}\sigma\, da' = \iint \vec{K} \times \vec{B} da' \qquad [3] \\ \vec{F} &= \iiint \vec{v} \times \vec{B}\rho d\tau' = \iiint \vec{J} \times \vec{B} d\tau' \end{aligned}$$

Where I is the current through a wire, **K** is the surface current density in Coulomb per second per meter, and **J** is the volume current density in Coulomb per second per square meter. Note that these equations include the following relations between charge densities and current densities:

$$\begin{aligned} \vec{I} &= \lambda\vec{v} \\ \vec{K} &= \sigma\vec{v} \qquad [4] \\ \vec{J} &= \rho\vec{v} \end{aligned}$$

We learned that the electrostatic and the magnetostatic force are related to each other by relativity. A current carrying conductor exerts a force on a moving charged particle that moves parallel to the wire. We explain this interaction by stating that the current carrying conductor creates a magnetic field around the wire and that a charge moving in a magnetic field feels a magnetic force. An observer that is moving on top of the charged particle though, will argue that there is no magnetic force as to him/her the charged particle has a zero velocity. He/she will still notice the force though, but claim that it is an electrostatic force. As he/she is moving parallel to the wire, he/she sees the electrons moving through the wire at a different speed as the observer in the laboratory frame. He/she also sees the positive ions in the wire move, in the opposite direction of the electrons. The different speeds between both observers will lead to different length contraction, and the distance between the positive and negative charges in the wire is no longer equal to each other. The observer riding on top of the charged particle will conclude that the wire has a net linear charge density unequal to zero, and because of this creates an electric field through space. Because of this electric field the charge feels a force.

Note that the equations stated above are only half of the story. One still needs to know the magnetic field in space in order to determine the magnetic force. Although one likes to think of it that magnetic fields are caused by moving charges, one cannot easily determine the magnetic field of a single moving charge. For one reason the length contraction which is at the root of the magnetic field is not defined for a single charge. It appears that a moving point charge will not create a steady magnetic field. So for magnetostatics we need a steady current. Steady currents are the sources of static magnetic fields. Furthermore we also assumed that the current has the same magnitude along the wire or that its divergence is equal to zero. This condition will guarantee that the charge density is constant independent of the time, so no changing electric fields either.

The magnetic field around a current distribution can be calculated from Biot-Savart's law:

$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} I \int \frac{d\vec{l}\,'\times\hat{r}_s}{r_s^2}$$
$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \iint \frac{\vec{K}(\vec{r}\,')\times\hat{r}_s}{r_s^2} da' \qquad [5]$$
$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \iint \frac{\vec{J}(\vec{r}\,')\times\hat{r}_s}{r_s^2} d\tau'$$

These equations work as long as the current distribution is known through whole space. We used those equations to derive the magnetic field caused by various current distributions:

1. Finite wire segment from θ_1 to θ_2:
$$\vec{B} = \frac{\mu_o I}{4\pi s}\left(\sin(\theta_2) - \sin(\theta_1)\right)\hat{\phi} \qquad [6]$$
2. Arc segment from –q to q (magnetic field in z-direction at a distance z above the center):
$$\vec{B} = \frac{\mu_o I}{2}\frac{R^2}{\left(R^2+z^2\right)^{3/2}}\frac{\theta}{\pi}\hat{k} \qquad [7]$$

We used Biot-Savart's law to derive Maxwell's equations for magnetism. Taking the divergence on both side of equation [5], we found that the divergence of B is equal to zero, i.e.

$$\nabla \bullet \vec{B} = 0 \qquad [8]$$

This equation implies that magnetic field lines close upon themselves, or in other words, the field lines are not sourced and sinked by magnetic charges (the magnetic monopole does not exist). Taking the curl on both sides of equation [5] results in the following expression:

$$\nabla \times \vec{B} = \mu_o \vec{J} \qquad [9]$$

We called this equation Ampere's law. We used it to determine the magnetic field around an infinite current carrying wire:

$$\vec{B} = \frac{\mu_o I}{2\pi s}\hat{\phi} \qquad [10]$$

Notice that the magnetic field varies inversely proportional to the distance between the field point P and the wire. We also used Ampere's law to determine the magnetic field above and below a plane that is carrying a surface charge density $K\hat{i}$:

$$\vec{B} = \begin{cases} \dfrac{\mu_o K}{2}\hat{j} & z < 0 \\ -\dfrac{\mu_o K}{2}\hat{j} & z > 0 \end{cases} \qquad [11]$$

Notice that the magnetic field around an infinite plane with a constant surface current density is independent of the distance of the field point to the plane. This result is similar as the result in electrostatics.

In addition to those two new Maxwell equations we also discussed the continuity equation. Charge is conserved so the net charge transport through an enclosed surface should correspond to the time derivative of the amount of charge within the volume enclosed by the surface, i.e.

$$\nabla \bullet \vec{J} = -\frac{\partial \rho}{\partial t} \qquad [10b]$$

We noticed that because of this equation, Ampere's law is not complete and might need an additional term. This additional term will reveal itself when we will introduce dynamic effects in chapter 7.

We compared typical electric fields in electrostatics with typical magnetic fields in magnetostatics. Positive electric charges are the sources of the electric fields and negative electric charges are the sinks. A positive point charge causes a star in the electric field. Note that the divergence of such field is only non-equal to zero at those positions were the point charges reside. So for an electric field around a point charge situated at the origin, the divergence is zero everywhere except for the origin.

Similarly the **B**-field has a non-zero curl for those points in space that have a non-zero current density. The electric field is however curl-less through whole space as long as we consider stationary fields

. Because the E-field is curl-less we can write the electric field as the gradient of a scalar field. We called this scalar field the electric potential:

$$\vec{E} = -\nabla V \qquad [12]$$

Note that there are many scalar fields for one certain electric field. If V is a solution of equation [12] then also V+constant is a solution of equation [12]

Currents are the sources of the magnetic fields. A current causes a twister in the magnetic field. The curl of this magnetic field is non-zero at those points in space where the current density is unequal to zero. So if we consider the magnetic field caused by a wire that is positioned along the z-axis, the curl of the magnetic field is only unequal to zero on the z-axis. Everywhere else in space the curl is zero. So the **B**-field has a non-zero curl for those points in space that have a zero current density. The magnetic field is divergence-less through whole space. As **B** is divergence-less a vector potential **A** can be defined so that:

$$\vec{B} = \nabla \times \vec{A} \tag{13}$$

Note that for a certain **B**-field one can find many vector fields **A** that with equation [13] all give the same magnetic vector field. In fact if **A** is a solution of equation [13] then also $\vec{A} + \nabla V$ is a solution of equation [13]. This is because the curl of a gradient is always equal to zero.

On page 235 in the book we proved that it is always possible to find an **A** for which the divergence of **A** becomes equal to zero. We choose this **A** for the magnetic vector potential because it simplifies Ampere's law:

$$\nabla \times \vec{B} = \nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \bullet \vec{A}\right) - \nabla^2 \vec{A} = -\nabla^2 \vec{A} = \mu_o \vec{J} \tag{14}$$

Note that equation [14] consists of 3 equations that are similar as the Laplace's equation from electrostatics. The electric potential V is replaced by a coordinate of **A**, and the charge density over ε_o is replaced by μ_o times a coordinate of **J**. It is clear from this that the sources of the magnetic vector potential are the current densities. If the current densities do not extend until infinity one can calculate A from J using the following equation:

$$\begin{aligned}
\vec{A}(\vec{r}) &= \frac{\mu_o}{4\pi} \iiint \frac{\vec{J}(\vec{r}\,')}{r_s} \, d\tau' \\
\vec{A}(\vec{r}) &= \frac{\mu_o}{4\pi} \oint \frac{I}{r_s} \, d\vec{l}\,' \\
\vec{A}(\vec{r}) &= \frac{\mu_o}{4\pi} \iint \frac{\vec{K}(\vec{r}\,')}{r_s} \, da'
\end{aligned} \tag{15}$$

A is called the magnetic vector potential. It is a weighted average of the current density. The currents farther away from a certain field point contribute less. The units of **A** are kgm/(Csec). Note that it is not a potential energy. It is sometimes referred to as a potential momentum. The field A shows up in the Hamiltonian when doing quantum mechanics. We will explain the meaning of **A** in a future chapter.

We used equation [15] to calculate the magnetic vector potential of various current distributions. For cases for which the current density extends all the way to infinity one cannot use equations [15]. In that case we need to use an equivalent of Ampere's law but now for **A**, ie.

$$\oint \vec{A} \bullet d\vec{l} = \iint \nabla \times \vec{A} \cdot d\vec{a} = \iint \vec{B} \bullet d\vec{a} = \Phi_M \qquad [16]$$

Note that this equation has the same mathematical form as Ampere's law in integral form. The enclosed current is however replaced by the enclosed magnetic flux, and **B** is replaced by **A**. We studied example 5.12.

We studied the boundary conditions of magnetostatics across a plane that carries a surface current density **K**, parallel to the plane. We found that the normal component of B was continuous across the interface. One of the tangential components (component of B perpendicular to the surface current but parallel to the interface) appears o be discontinuous. The other component appears to be continuous. We summarized the boundary conditions with the following vector equation:

$$\vec{B}_{above} - \vec{B}_{below} = \mu_o \vec{K} \times \hat{n} \qquad [17]$$

The vector potential is also continuous across the interface, i.e.

$$\vec{A}_{above} = \vec{A}_{below} \qquad [18]$$

The derivative of **A** perpendicular to the interface is however not continuous across the interface but proportional to the surface current density, i.e.

$$\frac{\vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_o \vec{K}$$

We did the same multipole expansion as in electrostatics for the magnetic vector potential A:

$$\vec{A}(\vec{r}) = \frac{\mu_o I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}\,' + \frac{1}{r^2}\oint r'\cos(\theta')d\vec{l}\,' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2(\theta') - \frac{1}{2}\right)d\vec{l}\,' + \ldots.\right] \qquad [19]$$

The first term in this multipole expansion, i.e. the monopole term, is always zero. This is because magnetic monopoles do not exists. So magnetic north and south poles come in pairs. This is different from the electrostatic multiplole expansion where the first term represents the net charge of the object. The dipole term can be rewritten to the following equation:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2} \qquad [20]$$

Where m is the magnetic dipole moment which is defined as:

$$\vec{m} = I\oint d\vec{a} = Ia \qquad [21]$$

The magnetic moment is a vector quantity, just like the electric dipole moment. For an ensemble of magnetic dipole moments, the total dipole moment can be calculated by taking the vector sum of the magnetic moments of the parts. We defined a pure magnetic dipole as a current distribution for which the multipole expansion will only result in a non-zero dipole term and all higher order terms are zero. Such current distribution consists of a current loop with an infinitesimal surface area and an infinite current, but a finite magnetic dipole moment Ia. For such pure magnetic dipole the magnetic vector potential is:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{m\sin(\theta)}{r^2}\hat{\phi} \qquad [22]$$

Note that A falls of with a one over square law. Note also that the magnetic vector potential is in the direction of ϕ. Equation [22] is the same as equation [21].

The magnetic field of such pure dipole can be calculated from [22] taking the curl:

$$\vec{B}(\vec{r}) = \frac{\mu_o m}{4\pi r^3}\left(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right) \qquad [23]$$

The magnetic field around a magnetic dipole falls of as $1/r^3$. This is similar as for an electric dipole, but considerably faster than for an electric charge. So the field of dipoles is more localized than the field of charges. This localization of the fields is important in magnetic recording. The more localized the field, the higher the possible information density.

Furthermore we learned that the magnetic force does not do work. Although that is very true and not always clear, this does not imply that you would not able to use magnetic fields to create devices and instrument that do work.