Summary chapter 6:

We defined the magnetic field to easier describe the relation between currents. In chapter 5 we learned that the force per unit length on a current carrying conductor (I ampere) that is exposed a magnetic field B is given by:

$$\frac{\vec{F}_{mag}}{l} = \vec{I} \times \vec{B} \qquad [1]$$

The focus of chapter 6 is on magnetic dipoles rather than wire segments. A magnetic dipole is closed wire loop. Its magnetic moment is given by:

$$m = AI \qquad [2]$$

Where A is the surface area of the loop and I the current in the loop. The direction of m is perpendicular to the surface area and can be found by the right hand rule. A magnetic dipole creates a magnetic field around itself that is given by the following expression (note that we assume that the dipole is in the origin and points towards the positive z-direction, the field point is in spherical coordinates):

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_o \vec{m}}{4\pi r^3}\left(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right) \qquad [3]$$

The corresponding magnetic vector potential of a magnetic dipole is given by the following expression:

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{m\sin(\theta)}{r^2}\hat{\phi} \qquad [4]$$

A magnetic dipole placed in a homogeneous magnetic field B will experience a torque, N, given by:

$$\vec{N} = \vec{m} \times \vec{B} \qquad [5]$$

If the magnetic field is inhomogeneous, the above equation will still give the correct torque as long as the magnetic dipole is a perfect dipole of infinitesimal size.

In a uniform field, the net force on a current loop is zero. In a non-uniform field this is no longer the case and a non-zero net force exists. For a perfect dipole **m** placed in a magnetic field **B**, the net force is:

$$\vec{F} = \nabla\left(\vec{m} \bullet \vec{B}\right) \qquad [7]$$

We also learned in section 6.1.3 that a magnetic field can speed up or slow down an electron orbiting a nucleus. The change in **m** is opposite to the change in **B** and results in diamagnetism.

We defined the magnetic dipole moment per unit volume as the magnetization of the material. The magnetization is identified by capital **M**. It is a vector quantity.

The field of a magnetized object can be calculated in two different methods. Assuming that the material consists of magnetic dipoles, one can add up the magnetic field of all those dipoles to find the magnetic field of the object. One normally first determines the magnetic vector potential by this method using the following expression:

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \iiint \frac{\vec{M}(\vec{r}\,') \times \hat{r}_s}{r_s^2} d\tau' \qquad [8]$$

Using integration by parts and introducing the bound volume current density and bound surface current densities we can rewrite equation [8] to:

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \iiint_V \frac{\vec{J}_b(\vec{r}\,')}{r_s} d\tau' + \frac{\mu_o}{4\pi} \oint\!\!\!\oint_S \frac{\vec{K}_b(\vec{r}\,')}{r_s} da' \qquad [9]$$

Where:

$$\vec{J}_b(\vec{r}\,') = \nabla \times \vec{M} \qquad [10]$$

$$\vec{K}_b(\vec{r}\,') = \vec{M} \times \hat{n} \qquad [11]$$

So rather than integrating over all magnetic dipoles, i.e. equation [8], one can integrate over all bound current contributions, i.e. equation [9]. The latter is simpler as the bound volume current densities are zero in a homogeneously magnetized object. In section 6.2.2 we learned the physical interpretation of the bound currents.

As one often does not know the bound currents we introduced the H-field. For the H-field one only has to consider the free currents, i.e.

$$\begin{aligned} &\nabla \times \vec{B} = \mu_o \vec{J} = \mu_o \left(\vec{J}_{free} + \vec{J}_{bound} \right) = \mu_o \left(\vec{J}_{free} + \nabla \times \vec{M} \right) \Leftrightarrow \\ &\nabla \times \left(\frac{1}{\mu_o} \vec{B} - \vec{M} \right) = \vec{J}_{free} \Leftrightarrow \nabla \times \vec{H} = \vec{J}_{free} \end{aligned} \qquad [12]$$

Or in integral form:

$$\oint \vec{H} \bullet d\vec{l} = I_{fenc} \qquad [13]$$

Be careful with equations [12] and [13]. The free currents are not the sole sources of the H-field. See also the handout on magnetic materials. So if there are no free currents, one cannot conclude from equation [12] and [13] that the magnetic H-field is zero, only that its curl is equal to zero. One can conclude though that the H field is curl-less in that particular case. This means that one can write the H-field as the gradient of a scalar field W, i.e.

$$\vec{H} = -\nabla W \tag{14}$$

As the divergence of B is zero one finds for the divergence of H:

$$\begin{aligned} &\nabla \bullet \vec{B} = \nabla \bullet \left(\mu_o\left(\vec{H} + \vec{M}\right)\right) = \mu_o \nabla \bullet \left(\vec{H} + \vec{M}\right) = 0 \Leftrightarrow \\ &\nabla \bullet \vec{H} = -\nabla \bullet \vec{M} \end{aligned} \tag{15}$$

Combining equations [14] and [15] gives:

$$\nabla^2 W = \nabla \bullet \vec{M} \tag{16}$$

Or in other words in that case the sources of the magnetic H-field are equal to those area that have a non-zero divergence. Note that the divergence is zero in a homogeneously magnetized object. Also note that the divergence is unequal to zero if the normal component of **M** has a non-zero component. Compare this reasoning with electrostatics where the divergence of **P** showed the location of the bound charges and the electric field within the material could be found by solving for Poisson's equation, i.e.

$$\nabla^2 V = \frac{\nabla \bullet \vec{P}}{\varepsilon_o} \tag{17}$$

We practiced solving equation [16] using the techniques of chapters 2 and 3 in problem 6.14 and 6.18.

Note that although W in magnetostatics has the same mathematical function as V in electrostatics, W does not give us the normalized potential energy. The meaning of W will become clear if one reverses equation [14], i.e.

$$W = -\int \vec{H} \bullet d\vec{l}$$

We found the following boundary conditions:

$$\begin{aligned} &H_{above}^{\perp} - H_{below}^{\perp} = -\left(M_{above}^{\perp} - M_{below}^{\perp}\right) \\ &\vec{H}_{above}^{//} - \vec{H}_{below}^{//} = \vec{K}_f \times \hat{n} \\ &B_{above}^{\perp} - B_{below}^{\perp} = 0 \\ &\vec{B}_{above}^{//} - \vec{B}_{below}^{//} = \mu_o \vec{K} \times \hat{n} \\ &W_{above} = W_{below} \end{aligned}$$