Laplace's equation Handout

Laplace's equation is given by:

$$\nabla^2 V = 0 \qquad [1]$$

In Cartesian coordinates this equation becomes:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad [2]$$

To solve this equation we use separation of variables. So we assume that we can write the solution of Laplace's equation, i.e. the electric potential function V(x,y,z), as the product of a function that only depends on x, i.e. X(x), and a function that only depends on y, i.e. Y(y), and a function that only depends on z, i.e. Z(z):

$$V(x, y, z) = X(x)Y(y)Z(z) \qquad [3]$$

I do not have a good explanation why the electric potential function can be separated in the three coordinates but some idea is provided by the classical monkey gun experiment where a marble is shot horizontally of a table and the same moment a 2nd marble is dropped of the table. Both marbles will hit the ground at the same moment from which we conclude that the motion in the x ad y-direction are independent of each other: so in mechanics we can resolve the force vectors in two orthogonal directions and treat the motions in both directions independent. Note that forces are basically nothing else than gradients of potential energies, i.e.

$$F = -\nabla U$$

Since the electric potential is nothing else than the potential energy of a test charge of 1 coulomb, we can assume that similarly to potential energy, the electric potential can be separated in three orthogonal directions, i.e. written as the product of three function (see equation [3]).

This is not a hard proof, and we might encounter problems for which this is not the case. But if we find a solution, then we can be sure that it is the solution since the uniqueness theorem shows that there is only one solution to Laplace's equation if we have the boundary conditions defined for all boundaries.

But if [3] is a solution of [2], then the2nd equal sign of [2] needs to be valid if we plug [3] into equation [2]. So let us do that. Plug [3] in [2] gives:

$$\frac{\partial^2 X(x)Y(y)Z(z)}{\partial x^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial y^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial z^2} = 0 \qquad [4]$$

As Y(y)Z(z) is a constant for the first derivative, it will factor out of the derivative. Similar for X(x)Z(z) for the 2nd derivative, and X(x)Y(y) for the third derivative. So we get:

$$Y(y)Z(z)\frac{\partial^2 X(x)}{\partial x^2} + X(x)Z(z)\frac{\partial^2 Y(y)}{\partial y^2} + X(x)Y(y)\frac{\partial^2 Z(z)}{\partial z^2} = 0 \qquad [5]$$

Divide the whole thing by X(x)Y(y)Z(z), gives:

$$\begin{gathered}\frac{Y(y)Z(z)}{X(x)Y(y)Z(z)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{X(x)Z(z)}{X(x)Y(y)Z(z)}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{X(x)Y(y)}{X(x)Y(y)Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = 0 \Leftrightarrow \\ \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = 0\end{gathered} \qquad [6]$$

Note that the last expression consists of a part that only depend on x, a part that only depends on y, and a part that only depends on z. By rearranging this expression we can separate equation [6] in three separate differential equations:

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} \qquad [7]$$

Note that the left side of [7] only depends on x and y and is independent of z, and the right side of [7] only depends on z and is independent of x and y. Since both sides are equal to each other each side is independent of x, y, and z. So each side of equation [7] should be a constant. Or in other words, equation [7] can be split into two equations:

$$\begin{gathered}\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -C_1 \\ -\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = -C_1\end{gathered} \qquad [8]$$

We can repeat this trick on the top equation of [8], i.e. rewrite this expression to:

$$\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -C_1 - \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} \qquad [9]$$

The left side of this expression only depends on x and is independent of y, and the right side of this equation only depends on y and is independent of x. As they are equal to each other both sides need to be independent of x and y. Or we can split [9] into two expressions:

$$-\frac{1}{X(x)}\frac{\partial^2 X(x)}{x^2}-C_1=C_2 \Leftrightarrow \frac{1}{X(x)}\frac{\partial^2 X(x)}{x^2}=C_3$$
$$\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{y^2}=C_2 \qquad [10]$$

Notice that the sum of C_1, C_2, and C_3 is equal to zero. So rather than having to solve equation [3] we now solve the following three differential equations:

$$\begin{aligned}
&\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}=C_3\\
&\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}=C_2\\
&\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}=C_1\\
&C_1+C_2+C_3=0
\end{aligned} \qquad [11]$$

Let us rewrite them:

$$\begin{aligned}
&\frac{\partial^2 X(x)}{\partial x^2}=C_3 X(x)\\
&\frac{\partial^2 Y(y)}{\partial y^2}=C_2 Y(y)\\
&\frac{\partial^2 Z(z)}{\partial z^2}=C_1 Z(z)\\
&C_1+C_2+C_3=0
\end{aligned} \qquad [12]$$

So we are looking for function for which the 2nd derivative is proportional to the function itself. I only know of two different type of solutions, i.e. trigonometric functions will have this property, and exponential functions have this property. Since you can write sine and cosine functions as the sum of two exponential functions (using the Euler equation) all solutions can be considered to be exponential.

To solve those equations we will use the following trial solutions:

$$X(x)=Ae^{\alpha x} \qquad [13]$$

Plugging into the first equation of [12] gives:

$$\frac{\partial^2 Ae^{\alpha x}}{\partial x^2}=C_3 Ae^{\alpha x} \Leftrightarrow \alpha^2 Ae^{\alpha x}=C_3 Ae^{\alpha x} \Leftrightarrow \alpha^2=C_3 \qquad [14]$$

The sign of C_3 will determine whether α is imaginary or real. In both cases we will have two solutions. So for negative C_3 we find:

$$\alpha = \pm i\sqrt{-C_3}$$

And thus the most general solution of [12a] is:

$$X(x) = A_1 e^{i\sqrt{-C_3}x} + A_2 e^{-i\sqrt{-C_3}x} \qquad [15]$$

Note that this is a trigonometric function. A_1 and A_2 are constants that still need to be determined. You will see in the examples of section 3.3. that we will use the boundary conditions to determine those values.

For positive C_3 we find:

$$\alpha = \pm\sqrt{C_3} \qquad [16]$$

And thus the most general solution of [12a] is:

$$X(x) = A_1 e^{\sqrt{C_3}x} + A_2 e^{-\sqrt{C_3}x} \qquad [17]$$

So the 2D or 3D Laplace's equation in Cartesian coordinates after separation of variables will have two type of solutions: (1) sinusoidal; (2) exponential. The type of solution in a certain direction depends on the sign of the separation constant. The sign of the separation can be determined from the boundary conditions provided in the problem. For Laplace problems, we will not repeat this whole derivation again and again. Instead we will determine the type of solution from the boundary conditions provided in the problem.

The two type of solutions have the following properties

1. Sinusoidal solution: i.e. the solution is periodic in one of the three space dimensions. Such solution has more than one zero:

   $$X(x) = A_1 \sin(kx) + A_2 \cos(kx) \qquad [18a]$$

   Note that a sinusoidal solution does not converge to zero for x goes to infinity. So problems in which one direction is not bounded such as for example 3.3 and that require the V to go to zero for x goes to infinity, will not have a sinusoidal form for that particular direction.

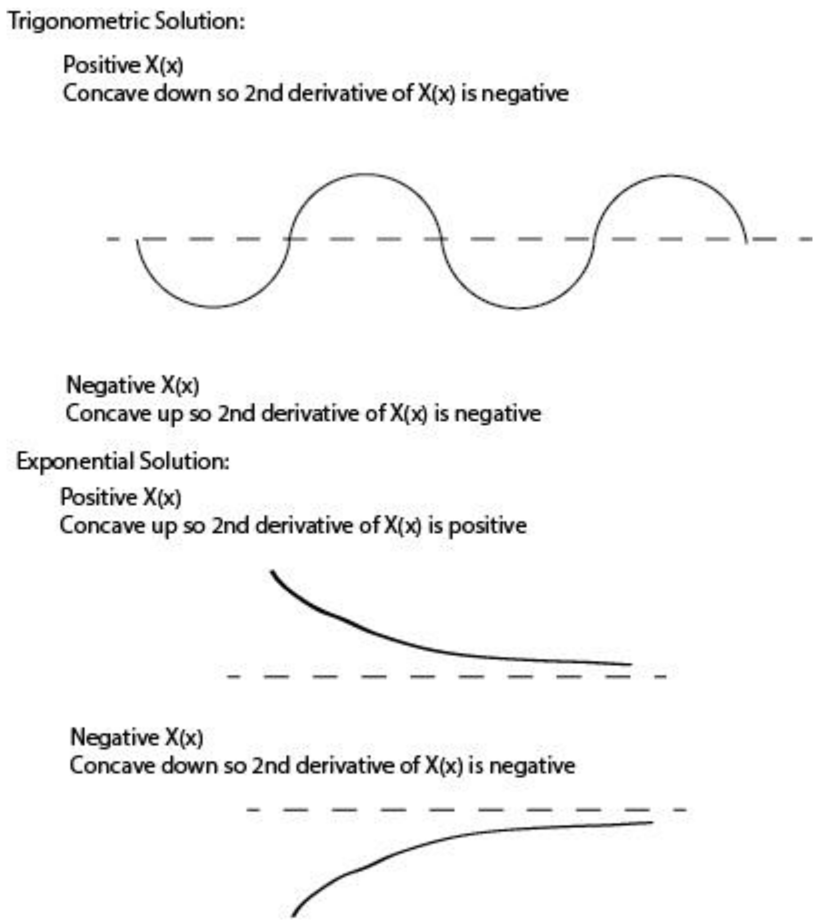
2. Exponential solution: i.e. the solution has no zero (both constants A_1 and A_2 have the same sign) or only one zero (both constants A_1 and A_2 have opposite sign).:

   $$X(x) = A_1 e^{kx} + A_2 e^{-kx} \qquad [18b]$$

   Note that an exponential solution has only one zero. So problems in which one direction has two zero boundary conditions require for that particular direction a sinusoidal solution.

The type of the solution depends on the sign of the separation constant (see above).

If the separation constant is positive, equation [12] tells us that the 2^{nd} derivative of the function has the same sign as the function itself. So the solution needs to be exponential. This is best understood by realizing that for positive values of X(x), X(x) has a positive 2^{nd} derivative (see equation [12] and the fact that C is positive). A positive 2^{nd} derivative means, the function is concave up. So if X(x) is positive and decreasing the function values go back to zero at a slower and slower rate (see Fig. 1 below). On the contrary if X(x) is negative, i.e. below the x-axis, the 2^{nd} derivative is also negative, so concave down. So if X(x) is increasing towards the X-axis, the increase will be slower and slower. Such solution is clearly exponential (see lower part of figure below).



If the separation constant is negative, equation [12] tells us that the 2^{nd} derivative has the opposite sign of the function itself. So if X(x) is positive, the 2^{nd} derivative of X(x) is negative, or is concave down. This implies that if X(x) is above the x-axis, it goes faster and faster back to the x-axis. If X(x) is negative, the 2^{nd} derivative must be positive, i.e. concave up. So if X(x) is below the x-axis, it goes faster and faster back to the x-axis. Clearly such solution is periodic (see figure above).

Note that not all three separation constants can have the same sign, because their sum needs to add up to zero (see equation 12). To decide which of the three directions has a positive separation constant and which has a negative separation constant one should look at the boundary condition. An exponential solution can only make one axis crossing. So any direction that has two zero boundary conditions needs to have a sinusoidal solution for that particular direction. Also directions for which the electric potential has to go to zero for points far away, needs to be exponential for that particular direction.

There is another way to understand why the solution in the three directions cannot be all sinusoidal. Such solutions will have local extremes; that is it will have an extreme that is not on the boundaries. Just think about it and assume that the solution in the x-direction is a half sine, the solution in the y-direction is a half sine, and the solution in the z-direction is a half sine. This will lead to a maximum in the middle of the box. Note that we started the chapter with the statement that solutions of Laplace equation have their extremes at the boundaries. So at least one of the dimensions must have an exponential solution.

If for some reason the boundary conditions reject both the trigonometric and the exponential solution, then we are out of luck, which means that we cannot use separation of variables to solve Laplace's equation.

The sinusoidal and exponential solutions provided in equations 18 and 19 come each with three constants that need to be determined. We determine those constants by using symmetry arguments and by using the boundary conditions in the x-direction, the boundary conditions in the y-direction, and the boundary conditions in the z-direction. So up to 6 boundary conditions depending on the dimension of the problem. Furthermore some of the constants in the various directions depend on each other because the sum of all separation constants need to be zero.

Note that some of the boundary conditions can lead to more than one solution. Look to example 3.3. The boundary conditions at y=0 and y=a lead to an infinite number of solutions of the form:

$$\sin\left(\frac{n\pi}{a}x\right) \text{ where n is a positive integer} \qquad [19]$$

Since Laplace's equation is a linear differential equation, the most general solution is a linear combination of all those solutions. So for example 3.3. applying the boundary conditions at y=0, y=a, and x is infinity will give the following solution:

$$V(x,y)=\sum_{n=1}^{\infty}C_n e^{n\pi x/a}\sin\left(\frac{n\pi\, y}{a}\right) \qquad [20]$$

Notice that one still has one more boundary condition at x=0. If we assume that the electric potential in the x=0 plane is equal to a constant V_o, one can apply the boundary condition by evaluating [20] at x=0 and setting that equal to the boundary condition at x=0:

$$V(0,y)=\sum_{n=1}^{\infty}C_n \sin\left(\frac{n\pi\, y}{a}\right)=V_o \qquad [21]$$

We use the Fourier trick and the last boundary condition (equation (21)) to find the remaining constants, Cn. We multiply both the left and right hand side of equation [21], with sin(n'πy/a) and then integrate over one period. You flip the sequence of the integral and the sum on the left side and then use the orthogonality property of the sine functions to simplify the left side, to one constant. The orthogonality property for sine functions is:

$$\int_0^a \sin\left(\frac{n\pi\, y}{a}\right)\sin\left(\frac{n'\pi\, \mathrm{y}}{\mathrm{a}}\right)dy = \begin{cases} 0 \text{ if } \mathrm{n}' \neq \mathrm{n} \\ \dfrac{a}{2} \text{ if } \mathrm{n}' = \mathrm{n} \end{cases}$$

More details on the Fourier trick are provided in the text.

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