## Summary chapter 4.

In chapter 4 dielectrics are discussed. In those materials the electrons are bonded to the atoms or molecules and cannot roam free through the material: "the electrons in insulators are on a tight leash and all they can do is move a bit within the atom or molecule. The cumulative effects of those microscopic displacements accounts for the characteristic behavior of dielectric materials.

Electric field s can distort charge distributions in atoms and molecules by stretching and rotating:

Although the center of positive charge and negative charge in atoms and most molecules that reside in a zero electric field coincides when placed in an electric field this is no longer the case. This results in polarization. We discussed atomic polarizability,  $\alpha$  of a single atom or a single molecule. It relates the electric dipole moment of the atom or molecule with the external electric field.

$$\vec{p} = a\vec{E}$$
 [1a]

Where **p** is the electric dipole moment. This is a vector pointing from the negative to the positive charge that has a magnitude equal to  $q^*d$ , where d is the distance between the positive and negative charge. We wrote the vector p is chapter 3 as:

$$ec{p}=q(ec{r}_{\scriptscriptstyle +}-ec{r}_{\scriptscriptstyle -})$$
 [1b]

Where  $\mathbf{r}_{+}$  ( $\mathbf{r}_{-}$ ) is the position vector of the positive charge (position vector of the negative charge).

We derived an equation for  $\alpha$  for a hydrogen atom:

$$\alpha = 4\pi\varepsilon_{a}a^{3}$$
[2]

Where a is the radius of the atom. Note that the typical polarizability is much smaller than the atomic radius (check example 4.1 and problem 4.2).

In addition we discussed molecules that have a permanent electric dipole moment. Such dipole moment originates from the different electronegativity of the atoms the molecule consists off. Electronegativity is a measure of the attraction of an atom for electrons in a covalent bond.

A permanent electric dipole experience a torque when placed in an uniform electric field. The torque about the center of the dipole is given by:

$$\vec{N} = \vec{p} \times \vec{E}$$
[3]

If the field is non-uniform the dipole also experiences a force:

$$\vec{F} = \left(\vec{p} \cdot \nabla\right)\vec{E} \tag{4}$$

This expression  $(\vec{p} \cdot \nabla)\vec{E}$  is the directional derivative of **E** in the direction of **p**, so how much does **E** change in the direction of **p**, or  $\Delta$ **E** in the direction of **p**. We discussed this one in chapter 1:

$$(\vec{p} \cdot \nabla)\vec{E} = \left( p_x \frac{\partial E_x}{\partial x} + p_y \frac{\partial E_x}{\partial y} + p_z \frac{\partial E_x}{\partial z} \right)\hat{i} + \left( p_x \frac{\partial E_y}{\partial x} + p_y \frac{\partial E_y}{\partial y} + p_z \frac{\partial E_y}{\partial z} \right)\hat{j} + \left( p_x \frac{\partial E_z}{\partial x} + p_y \frac{\partial E_z}{\partial y} + p_z \frac{\partial E_z}{\partial z} \right)\hat{k}$$

$$(5)$$

Note that equation [3] is also valid for a non-uniform electric field as long as one is interested in calculating the torque about the center of the dipole. The torque about any other point **r** is given by:

$$\vec{N} = \left(\vec{p} \times \vec{E}\right) + \left(\vec{r} \times \vec{F}\right)$$
[6]

As objects consists of a lot of permanent or induced electric dipoles we would like to use the electric dipole density. This quantity is called the polarization and is identified by a capital **P**. Note that this is a vector quantity. Its magnitude tells us the electric dipole moments per unit of volume. You can compare it with the charge density  $\rho$  we introduced in chapter 2. Of course the charge density is not a vector but a scalar quantity.

The field of a polarized object is the summation of the fields caused by all electric dipoles the object consists of. The electric potential of a single dipole was derived and discussed in chapter 3 (equation 3.102):

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \frac{\vec{r}_{script} \cdot \vec{p}}{r_{script}^2}$$
[7]

So for a polarized object:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \int_{volume} \frac{\vec{r}_{script} \cdot \vec{P}(\vec{r}\,') d\tau'}{r_{script}^2}$$
[8]

Note that from equation [7] to [8] we replaced the electric dipole moment, **p**, by the product of the polarization and a volume element, i.e.  $Pd\tau$ . Equation [7] corresponds to equation 4.9 in the text. As calculating equation [8] is time consuming we simplified equation [8] to:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \int_{volume} \frac{\rho_b}{r_{script}} d\tau' + \frac{1}{4\pi\varepsilon_o} \int_{surface} \frac{\sigma_b}{r_{script}} da'$$
[9]

Where  $\rho_p$  is the bound volume charge density and  $\sigma_b$  is the bound surface charge density. They are given by:

$$\rho_b = -\nabla \cdot \vec{P}$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$
[10]

So rather than have to sum up over all contributions of the electric dipoles, we can sum up over all bound charges. In class we discussed the physical interpretation of bound charges. The displacement of the charges in the material caused by the polarization will only result in a net charge near the surface of the material or near areas where the polarization is not constant, i.e. has a non-zero divergence.

Note that there are two ways to determine the electric field. One can first determine the electric potential using equation [9] and then calculate the electric field from the electric potential using:

$$\vec{E} = -\nabla V \tag{11}$$

Or one can use Coulomb's law and integrate over all charges. If no free charges are there, this will be an integration over the  $\rho_b$  and  $\sigma_b$ . So for this approach the problem converts to the problems we discussed in chapter 2.

Of course there is a third method to find E, but that one is not very easy and not discussed in the text. We could start off with the electric field of a single dipole, i.e. equation 3.103, and then integrate over all dipoles of the object. The math will be horrendous .....

So normally we first determine the bound charge using equation [10], then use equation [9] to find V, and then use equation [11] to find E if asked for. For symmetric problems we might want to use Gauss' law and directly determine E instead of equations [9] and [11]. Note that Gauss' law for E takes into consideration all charges, i.e. the bound and the free charges. As the electric field in the object is often not constant, also the polarization in the object varies as a function of the position. This makes it often difficult to determine the bound charges. We therefore introduced a new parameter, the electric displacement D. This is also a vector quantity and is defined as:

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$
[12]

The reason for this introduction becomes clear once we take the divergence of equation [12], i.e.

$$\nabla \cdot \vec{D} = \varepsilon_o \nabla \cdot \vec{E} + \nabla \cdot P = \varepsilon_o \nabla \cdot \vec{E} - \rho_b$$
[13]

Note that the divergence of **E** is equation to  $\rho/\epsilon_o$ , where  $\rho$  is the total charge density i.e. bound charge density and free charge density (this is Gauss' law in differential form, i.e. equation 2.14), so we can rewrite equation [13] now to:

$$\nabla \cdot \vec{D} = \rho_{free} \tag{14}$$

This Gauss' law for the electric displacement. It says that the divergence of **D** is equal to the free charge density. In integral form this law becomes:

$$\oint_{surface} \vec{D} \cdot d\vec{a} = Q_{\text{free enclosed}}$$
[15]

Note that on the left side of equations [14] and [15] are no  $\varepsilon_o$ . For linear dielectric materials, we can use equation [14] and [15] to first determine the **D**. The advantage of such approach is that we do not need to know the bound charge volume and surface charge density as only the free charges are to be known. This simplifies the math considerable. Note that for the capacitor problems in chapter 4 we first determine **D** from the free charges and then use the following equation to determine **D** from **E**:

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} = \varepsilon_o \vec{E} + \varepsilon_o \chi_e \vec{E} = \varepsilon_o (1 + \chi_e) \vec{E} = \varepsilon_o \varepsilon_r \vec{E} = \varepsilon \vec{E}$$
[16]

Note that this equation implied a lot of relations. I would like you to memorize those relations, and they should also be on your equation sheet.

E0 is called the permittivity of free space,  $\varepsilon$  is called the permittivity of the material,  $\varepsilon_r$  is called the dielectric constant of the material ( $\varepsilon/\varepsilon_o$ ),  $\chi_e$  is called the electric susceptibility of the material and for linear dielectric materials is defined by:

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$
[17]

Compare equation [17] with equation [1] and you have to conclude that the electric susceptibility,  $\chi_e$ , is related to the atomic polarizability,  $\alpha$ . There is a caveat though, as the E in equation [1] is the external applied field, and the E in equation [17] is the total field felt by the dipole, i.e. the effect of the external electric field and the induced electric field caused by the bound charges. The latter is often opposite to the external electric field. So the relation between the electric susceptibility and the atomic polarizability is not straight forward and will be discussed in solid state.

We discussed the boundary conditions for dielectric materials:

$$D_{above}^{\perp} - D_{below}^{\perp} = \sigma_{free}$$

$$\vec{E}_{above}^{\prime\prime} - \vec{E}_{below}^{\prime\prime} = 0$$
[18]

If there is no free charges, i.e.  $\sigma_f=0$ , the perpendicular component of **D** will be continuous and the parallel component of E will also be continuous across the interface. Notice that the electric potential, i.e. V will also be continuous across the interface. The first equation can be derived from equation [14], i.e. Gauss' law for D, and the 2<sup>nd</sup> equation can be derived from the fact that E is curl-less.

The energy of a dielectric system can be calculated from the energy density equation, i.e. :

energy density 
$$=\frac{1}{2}\vec{D}\cdot\vec{E}$$
 [19]

So the total energy of the system

$$W = \frac{1}{2} \iiint_{volume} \vec{D} \cdot \vec{E} d\tau$$
[20]

Note that the energy density is not always constant through space and that one might need to find separate expressions for the energy density inside and outside the material and then do the integral in two steps. Equation [20] includes the electrostatic energy stored in the free charges, the electrostatic energy stored in the bound charges, and the spring energy stored in the bound charges. The latter two are opposite and equal in magnitude.

At the end of chapter 4 we learned how to calculate forces from the total energy expression. As the energy is lowered when one places a dielectric material inside a capacitor, a force is acting on the dielectric material, that sucks the dielectric material in between the plates. We derived an expression on how to determine the force from the capacitance and the electric potential across the capacitor plates:

$$F = \frac{1}{2}V^2 \frac{dC}{dx}$$
[21]

We derived this expression for the situation that the electric charge on the capacitor plates is constant. The expression is however also valid for the case that the electric potential difference across the plates is kept constant.