To make current flow you have to push on the charges. For most materials:

$$\vec{J} = \sigma \vec{E} = \frac{1}{\rho} \vec{E}$$
^[1]

The resistivity is a parameter that varies more than 24 orders of magnitude between silver (1.6E-8 Ohm.m) and fused quartz (1E16 Ohm.m).

Because of this linear relation between current density and electric field there is also a linear relation between current through a device (I) and voltage across (V) the terminals of a device. To double the voltage, you double the charge which doubles the **E**, which double the **J**, which doubles the I.

Note that conductivity and resistivity are materials parameters while resistance (R) and conductance (G) are device parameters:

$$V = IR$$

$$I = GV$$
[2]

We call this Ohm's law also it is not a real law much more a rule of thumb. We derived a relation between current density and electric field using a microscopic model, calculating first the acceleration from the applied electric field, turning that into an acceleration of the electrons. We furthermore assumed that the drift velocity of the electrons is proportional to the mean free path between two collisions of an electron. The mean free path is considered to be a materials property and depend on the concentration of scatter centre (collisions with grain boundaries, defects, impurities, or lattice vibrations (phonons)). As the thermal velocity is much larger than the drift velocity, the mean time between two collisions is independent of the of the drift velocity of the electrons, and the average drift velocity is linear dependent on the acceleration and thus the applied electric field. We found:

$$\vec{J} = nfq\vec{v}_{drift} = \frac{nf\lambda q^2}{2mv_{thermal}}\vec{E}$$
[3]

Where n is the number of atoms per unit of volume, f the number of free electrons per atom, q the charge of an electron, m the effective mass of an electron, and λ the mean free path of the electrons. So electrons will have a large thermal velocity and then a much smaller drift velocity. Applying an electric field to a material will accelerate the free electrons until the free electrons collide with an imperfection in the material (think in terms of crystal structure) which will reset the drift velocity back to zero. At the collision the kinetic energy of the electrons in converted into heat, i.e. vibration of the atoms:

$$P = VI = I^2 R$$
^[4]

We studied the I-V relation of devices with different geometry (Problem 7.1), i.e. the conventional rectangular resistor, the coaxial cable, and spherical device. To find the I-V relation we assume a charge on the electrodes, use Gauss' law to determine the electric field in the device, and then use equation [1] and the relation between J and I to find the current going through the device by integrating over an appropriate cross sectional area:

$$I = \iint \vec{J} \bullet d\vec{a} = \iint \sigma \vec{E} \bullet d\vec{a}$$
^[5]

Note that this approach is very similar to the method used to determine the capacitance of a device in chapter 2 (example 2.11). Example 7.3 shows that for rectangular devices with planar electrodes the electric field is homogeneous through the device. We also looked to other geometries, i.e. P7.39 and 7.40.

We defined electromotive force of a circuit loop:

$$\mathcal{E} = \oint \vec{f} \bullet d\vec{l}$$
[6]

Where f is the total force per unit charge on a positive charge consisting of two parts, i.e. an electrostatic part **E** and a pushing part of the source f_s. The latter contains also any emf generated by a varying magnetic flux caught by the loop. In a source E=-fs, so the voltage across the terminals of a battery is:

$$V = -\int_{a}^{b} \vec{E} \bullet d\vec{l} = \int_{a}^{b} f_{s} \bullet d\vec{l} = \mathcal{E}$$
[7]

Although batteries and voltage and current sources create a local non zero emf, the emf caused by a flux change through the loop is distributed across the loop. We learned that a changing magnetic field through the loop, or moving the loop away from the magnetic field, or changing the size of the loop all result in an induced emf in the loop. We defined a flux rule:

$$\mathcal{E} = -\frac{d\Phi}{dt}$$
[8]

And from this flux rule we derived Faraday's law in integral and differential form:

$$\oint \vec{E} \bullet d\vec{l} = -\iint \frac{\partial \vec{B}}{\partial t} \bullet d\vec{a}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
[9]

We studied various inductors, i.e. solenoid and torroid, and noticed that a changing current through a system of wires result in a changing magnetic field and thus results in an electric field, i.e. induced emf. According to Ampere's law B is proportional to current. The proportional constant for a device was

called the inductance of the device. We defined two different inductances, i.e. the self-inductance L, to describe the relation between the current through the device and the flux generated by the device:

$$\Phi = LI$$
[10]

And the mutual inductance, M, to describe the flux coupling between two devices:

$$\Phi_2 = M_{21}I_1 \tag{11}$$

We derived an expression to calculate the flux through a device from the magnetic vector potential A:

$$\Phi_2 = \iint \vec{B}_1 \bullet d\vec{a}_2 = \oint \vec{A}_1 \bullet d\vec{l}_2$$
[12]

Note that we have seen this relation before in chapter 5, i.e. example 5.12. This relation can be used to derive an expression for the mutual inductance of a device:

$$M_{21} = \frac{\mu_o}{4\pi} \oint \oint \frac{d\vec{l}_1 \bullet d\vec{l}_2}{\vec{r}_s}$$
[13]

This is called the Neumann formula, although not so useful for most inductance problems, it shows that $M_{21}=M_{12}$, and furthermore it shows that M only depends on the geometry of the device.

We studied how to determine the self inductance or mutual inductance of a set of wires, i.e. first determine B from the current using Ampere's law and then integrating the B across a suitable cross-sectional area to find the flux (example 7.10). We also derived an equation for the induced emf of an inductor:

$$\mathcal{E} = -L\frac{dI}{dt}$$
[14]

Which with the proper sign convention (i.e. voltage drop in the direction of the current similar to that of a resistor) will result in the following I-V relation for an inductor:

$$V = L \frac{dI}{dt}$$
[15]

An expression for the energy stored in an inductor was derived from equation [4]. We have seen this expression in PHYS2425:

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi$$
 [16]

Substituting the inductor version of [12] in [16] and rewriting gives and alternative expression for the magnetostatic energy:

$$W = \frac{1}{2\mu_o} \left[\iiint_V B^2 d\tau - \oiint_S \left(\vec{A} \times \vec{B} \right) \bullet d\vec{a} \right]$$
[17]

If the integral is taken over the whole volume the surface integral becomes zero as **A** and **B** are going to zero at large distance from the source currents. So this results in two different magnetostatic energy expressions:

$$W = \frac{1}{2\mu_o} \iiint_{all \text{ space}} B^2 d\tau$$

$$W = \frac{1}{2} \iiint_{all \text{ space}} \vec{A} \bullet \vec{J} d\tau$$
[18]

For the 2nd equation one often only integrates over the area where the current density is unequal to zero.

We found that a term needs to be added to Ampere's law for Maxwell's equations to be consistent with the continuity equation. The total set of Maxwell's equation in differential form become:

$$\nabla \bullet \vec{E} = \frac{1}{\varepsilon_o} \rho$$

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{J} + \varepsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}$$
[19]

Those equations together with the force law:

$$\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right)$$
[20]

summarize the entire theoretical content of classical electrodynamics and even imply the continuity equation i.e.

$$\nabla \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$
[21]

Maxwell's equations in matter are expressed in terms of the auxiliary fields, $D=\epsilon_0 E+P$ and $H=1/\mu_0 B-M$, i.e.

$$\begin{aligned} \nabla \bullet \vec{D} &= \rho_{free} \\ \nabla \bullet B &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t} \end{aligned}$$
 [22]

The 2nd term in the Ampere Maxwell equation is called the displacement current density, this is also a bound current density but should not be confused with the bound current densities originating from the magnetization we discussed in chapter 5, i.e

$$\vec{J}_{b} = \nabla \times \vec{M}$$

$$\vec{K}_{b} = \vec{M} \times \hat{n}$$
[23]

Note that we arrived at these Maxwell equations after substituting the charges and current densities in equation [19] by:

$$\rho = \rho_B + \rho_{free} = -\nabla \bullet P + \rho_{free}$$
[23a]

$$\vec{J} = \vec{J}_{free} + \vec{J}_{bound} + \vec{J}_{polarization} = \vec{J}_{free} + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$
[23b]

Note that if the polarization of a cube is varying as a function of the time the bound surface charge at the top and bottom of the cube increases. Associated with this bound surface charge is a current. We refer to this current as the polarization current and it originates from the electric dipole moment to be time dependent. Such a time dependent polarization will result in a net transport of charge from the bottom to the top surface of the cube (assuming polarization is in the up direction in the cube).

For linear materials the constitutive equations are:

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$
[24]

Or

→ →

$$D = \varepsilon E$$

$$\vec{H} = \frac{1}{\mu} \vec{B}$$
[25]

The boundary conditions for the em field were summarized in section 7.3.6:

$$\vec{D}_{1} \bullet \vec{a} - \vec{D}_{2} \bullet \vec{a} = \sigma_{free} a$$

$$\vec{B}_{1} \bullet \vec{a} - \vec{B}_{2} \bullet \vec{a} = 0$$

$$\vec{E}_{1}^{''} - \vec{E}_{2}^{''} = 0$$

$$\vec{H}_{1}^{''} - \vec{H}_{2}^{''} = \vec{K}_{free} \times \hat{n}$$
[26]