Key Homework 7.1

1. Consider two concentric metal cylindrical shells of 1 meter length and of radius a and b, respectively. They are separated by a weakly conducting material of conductivity σ.
   a. If they are maintained at a potential difference V, what current flows from one to the other.
   b. What is the resistance between the shells?

a. This problem is similar to problem 7.1 in the text but has a different geometry. To determine the electric field assume that the inner cylinder contains a charge of Q Coulomb/m. Now use Gauss' law to determine the electric field in the space between both cylinders. As Gaussian surface we use a cylinder with a radius r which is larger than a but smaller than b.

$$\oiint_S \vec{E} \bullet d\vec{a} = E 2\pi r l = \frac{1}{\varepsilon_o} q_{enclosed} = \frac{Ql}{\varepsilon_o} \Leftrightarrow$$

$$\vec{E} = \frac{Q}{\varepsilon_o 2\pi\, r} \hat{\phi}$$

The current can now be calculated from J, i.e.

$$J = \sigma E = \frac{\sigma Q}{\varepsilon_o 2\pi\, r}$$

And we find I by integration, i.e. for a piece of l meter the I is:

$$I = \iint_S \vec{J} \bullet d\vec{a} = \frac{\sigma Q}{\varepsilon_o 2\pi\, r} 2\pi r l = \sigma \frac{Q}{\varepsilon_o} l$$

b. Now we can use the E-V relation of chapter 2 (equation 2.21) to determine Vab:

$$V_a - V_b = -\int_b^a \frac{Q}{\varepsilon_o 2\pi\, r} dr = \frac{Q}{2\pi\varepsilon_o} \ln\left[\frac{b}{a}\right]$$

Now we calculate R from V and I, for an l meter length:

$$R = \frac{V}{I} = \frac{\ln\left[\frac{b}{a}\right]}{2\pi\, l}$$

2. Work problem 7.2 in the text, i.e. the RC circuit in the text on page 302.

Problem 7.2

(a) $V = Q/C = IR$. Because positive I means the charge on the capacitor is *decreasing*, $\frac{dQ}{dt} = -I = -\frac{1}{RC}Q$, so $Q(t) = Q_0 e^{-t/RC}$. But $Q_0 = Q(0) = CV_0$, so $\boxed{Q(t) = CV_0 e^{-t/RC}.}$

Hence $I(t) = -\frac{dQ}{dt} = CV_0 \frac{1}{RC} e^{-t/RC} = \boxed{\frac{V_0}{R} e^{-t/RC}.}$

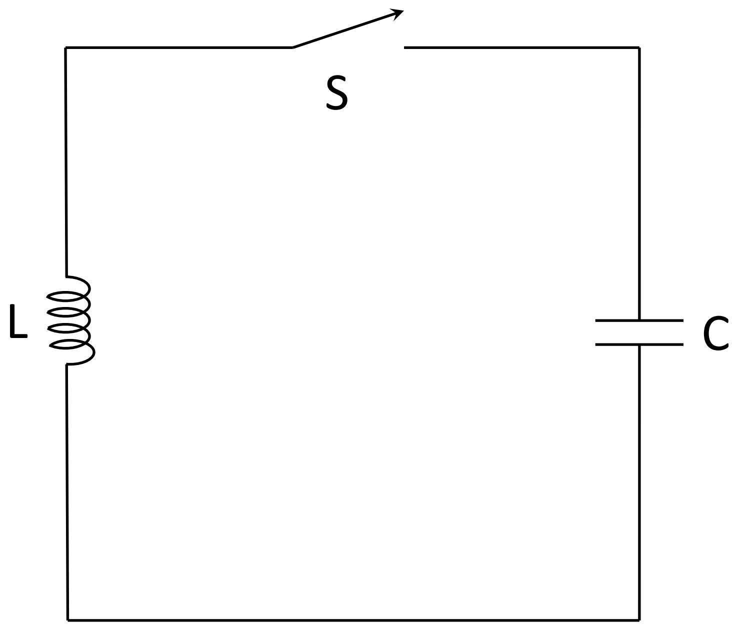
(b) $W = \boxed{\frac{1}{2}cV_0^2.}$ The energy delivered to the resistor is $\int_0^\infty P\,dt = \int_0^\infty I^2 R\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC}\,dt = \frac{V_0^2}{R}\left(-\frac{RC}{2}e^{-2t/RC}\right)\Big|_0^\infty = \frac{1}{2}CV_0^2.$ ✓

(c) $V_0 = Q/C + IR$. This time positive I means Q is *increasing*: $\frac{dQ}{dt} = I = \frac{1}{RC}(CV_0 - Q) \Rightarrow \frac{dQ}{Q - CV_0} = -\frac{1}{RC}dt \Rightarrow \ln(Q - CV_0) = -\frac{1}{RC}t + \text{constant} \Rightarrow Q(t) = CV_0 + ke^{-t/RC}$. But $Q(0) = 0 \Rightarrow k = -CV_0$, so $\boxed{Q(t) = CV_0\left(1 - e^{-t/RC}\right).}$ $I(t) = \frac{dQ}{dt} = CV_0\left(\frac{1}{RC}e^{-t/RC}\right) = \boxed{\frac{V_0}{R}e^{-t/RC}.}$

(d) Energy from battery: $\int_0^\infty V_0 I\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-t/RC}\,dt = \frac{V_0^2}{R}\left(-RCe^{-t/RC}\right)\Big|_0^\infty = \frac{V_0^2}{R}RC = \boxed{CV_0^2.}$

Since $I(t)$ is the same as in (a), the energy delivered to the resistor is again $\boxed{\frac{1}{2}CV_0^2.}$ The final energy in the capacitor is also $\boxed{\frac{1}{2}CV_0^2,}$ so $\boxed{\text{half}}$ the energy from the battery goes to the capacitor, and the other half to the resistor.

3. A capacitor C is charged up to a potential V and connected to an inductor L as shown below. At time $t = 0$, the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with L and C?



The induced voltage across the inductor must be equal to the voltage across the capacitor. The sign of this equation however depends on the direction of the current. If the current flows towards the positive plate of the capacitor the sign should be negative, if the current flows away from the positive plate of the capacitor the sign is positive.

$$L\frac{dI}{dt} = \frac{Q}{C}$$

We can make this more manageable by realizing that the charge on the capacitor is related to the derivative of the current through the inductor. The sign of this equation is negative if the current definition is away from the positive pole of the capacitor but the sign of the equation is positive if the current definition in the circuit is towards the positive pole of the capacitor.

$$I = -\frac{dQ}{dt}$$

The equation then becomes

$$-L\frac{d^2Q}{dt^2} = \frac{Q}{C}$$

Or,

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

This is a second order differential equation for Q that is completely analogous to the equation of motion for a simple harmonic oscillator. The solution is

$$Q = Acos(\omega t) + Bsin(\omega t)$$

Where

$$\omega = \frac{1}{\sqrt{LC}}$$

At $t = 0$, we know that $Q = CV$, which means $A = CV$. The current is then given by

$$I = \frac{dQ}{dt} = -CV\omega sin(\omega t) + B\omega cos(\omega t)$$

We know $I = 0$ at $t = 0$, which means $B = 0$. The current in the circuit is therefore given by

$$\boxed{I(t) = -V\sqrt{\frac{C}{L}} sin\left(\frac{t}{\sqrt{LC}}\right)}$$

If a resistor is added, the differential equation then becomes

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$$

This is analogous to the equation of motion for a damped harmonic oscillator, so the solution would look like the velocity of a damped oscillator. The actual solution (underdamped, overdamped, or critically damped), would depend on the relative sizes of L, R, and C.

4. Work problem 7.3 in the text on page 302.

Problem 7.3

(a) $I = \int \mathbf{J} \cdot d\mathbf{a}$, where the integral is taken over a surface enclosing the positively charged conductor. But $\mathbf{J} = \sigma \mathbf{E}$, and Gauss's law says $\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q$, so $I = \sigma \int \mathbf{E} \cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0} Q$. But $Q = CV$, and $V = IR$, so $I = \frac{\sigma}{\epsilon_0} CIR$, or $\boxed{R = \dfrac{\epsilon_0}{\sigma C}.}$ qed

(b) $Q = CV = CIR \Rightarrow \frac{dQ}{dt} = -I = -\frac{1}{RC} Q \Rightarrow \boxed{Q(t) = Q_0 e^{-t/RC}}$, or, since $V = Q/C$, $V(t) = V_0 e^{-t/RC}$. The time constant is $\tau = RC = \boxed{\epsilon_0/\sigma.}$