The differential Maxwell's equations:

$$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} \qquad [1]$$

These equations are very useful to calculate the field on the right side from the field on the left side. To go from right to left it is often better to use the integral form of the equations. We can convert from differential to integral form via the following recipes:

1. The first two Maxwell equations that contain a divergence on the left side, can be converted in the integral form by taking the volume integral on both sides and converting the left side with the divergence theorem from a volume integral to a surface integral.
2. The last two Maxwell equations that contain a curl on the left side, can be converted in the integral form by taking the surface integral on both sides and converting the left side with Stokes theorem form a surface integral to a line integral.

The integral Maxwell's equations:

$$\iiint_V \nabla \bullet \vec{E} d\tau = \iiint_V \frac{\rho}{\varepsilon_0} d\tau \therefore \oiint_S \vec{E} \bullet d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_o}$$

$$\iiint_V \nabla \bullet \vec{B} d\tau = 0 \therefore \oiint_S \vec{B} \bullet d\vec{a} = 0$$

$$\iint_S \nabla \times \vec{E} \bullet d\vec{a} = -\iint_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{a} \therefore \oint_K \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t} \iint_S \vec{B} \bullet d\vec{a} \therefore \oint_K \vec{E} \bullet d\vec{l} = -\Phi_B \qquad [2]$$

$$\iint_S \nabla \times \vec{B} \bullet d\vec{a} = \iint_S \mu_o \vec{J} \bullet d\vec{a} + \mu_o \varepsilon_o \iint \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a} \therefore \oint_S \vec{B} \bullet d\vec{l} = \mu_o I_{enclosed} + \mu_o \varepsilon_o \frac{\partial}{\partial t} \Phi_E$$

To go from the fields on the right to the fields on the left we often use the integral Maxwell's equations and look for symmetry so for the left integrals the field can be factored out of the integral expression.

In addition we defined two auxiliary fields, i.e. the electrostatic potential and magnetic vector potential:

$$\vec{E} = -\nabla V$$

$$\vec{B} = \nabla \times \vec{A} \qquad [3]$$

Note that V and **A** are not unambiguously determined by those equations. As there are many different scalar fields that have the same gradient we agreed for most of the problems to take the V that is zero very far away from the charges, i.e. in infinity. For **A** we agreed to choose the vector field that has a zero divergence. Writing Gauss' law in terms of V gives, Laplace's equation, i.e.

$$\nabla^2 V = \frac{\rho}{\varepsilon_0} \qquad [4]$$

Writing Ampere's law with Maxwell's extension in terms of A gives again Laplace's equation, i.e.

$$\nabla^2 \vec{A} = \mu_o \vec{J} + \varepsilon_o \mu_o \frac{\partial \vec{E}}{\partial t} \qquad [5]$$

Note that this last equation is actually a set of three Laplace's equations, one for each coordinate of the vector. Note also the 2nd term on the right of equation [5] often dubbed as the displacement current.

Although we discussed the boundary conditions of the E and B fields at several places in the text and I asked you to memorize some of them, I often feel it is easier to memorize Maxwell's equations in differential form and then derive the boundary conditions whenever I need them. The first two Maxwell's equations tell me something about the normal component of the fields across a planar interface, and the latter two Maxwell's equations about the parallel components of the fields across a planar interface. The rule is think pill-box for the div and Amperian loop for curl.

When dealing with materials we defined two more auxiliary fields, i.e. magnetic H-field (**H**) and electric displacement (**D**):

$$\begin{aligned} \vec{H} &= \frac{1}{\mu_o}\vec{B} - \vec{M} \\ \vec{D} &= \varepsilon_o \vec{E} + \vec{P} \end{aligned} \qquad [6]$$

Note that M is the volume magnetic dipole moment density and P is the volume electric dipole moment density. When we substitute these equations in the Maxwell's equations we can derive Maxwell's equations for materials, i.e.

$$\begin{aligned} \nabla \bullet \vec{D} &= \rho_{free} & \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \bullet \vec{B} &= 0 & \nabla \times \vec{H} &= \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t} \end{aligned} \qquad [7]$$

Note that with these auxiliary fields the charge and current sources are referring to the free charges and free currents. So **D** and **H** are the fields corrected for the bound charges ($\rho_b = \nabla \bullet \vec{P},\ \sigma_b = \vec{P} \bullet \hat{n}$) and the bound currents ($\vec{J}_b = \nabla \times \vec{M}, \mathrm{K}_b = \vec{M} \times \hat{n},\ \mathrm{J}_P = \frac{\partial \vec{P}}{\partial t}$). Although I could derive equations [7] from [6] and [1], I find it easier to just memorize those. Boundary conditions at planar interfaces in materials I just derive from [7] using the first two Maxwell's equations to determine what happens to the perpendicular component of the **D** and **B** fields, and the last two Maxwell's equation to determine what happens to the parallel component of **E** and **H** across a planar interface.

We learned that energy is stored in Electric and Magnetic fields:

$$W_B = \frac{1}{2\mu_o}\iiint B^2 d\tau = \frac{1}{2}\iiint \vec{A} \bullet \vec{J} d\tau \qquad W_E = \frac{\varepsilon_o}{2}\iiint E^2 d\tau = \frac{1}{2}\iiint V\rho d\tau \qquad [8]$$