

Key HW 8_1

Concept Question: Capacitor

The figures above show a side and top view of a capacitor with charge Q and electric and magnetic fields E and B at time t . At this time the charge Q is:

1. Increasing in time
2. Constant in time.
3. Decreasing in time.

Problem: Capacitor

A circular capacitor of spacing d and radius R is in a circuit carrying the steady current I shown. At time $t=0$ it is uncharged

1. Find the electric field $E(t)$ at P vs. time t (mag. & dir.)
2. Find the magnetic field $B(t)$ at P
3. Find the Poynting vector $S(t)$ at P
4. What is the total power flux into/out of the capacitor?
5. Does this make sense? How? (Hint: What's U ?)

1. Increasing as the Poynting vector points into the capacitor; the direction of B implies that dE/dt is in the same direction as E , so increasing as Ampere's-Maxwell's law does not have the minus sign.
2. Use Gauss' law to determine the electric field:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \Leftrightarrow E\pi R^2 = \frac{It}{\epsilon_0} \Leftrightarrow \vec{E} = \frac{It}{\epsilon_0 \pi R^2} \hat{z}$$

Use Ampere-Maxwell's law to determine the magnetic field:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \iint \vec{J} \cdot d\vec{a} + \mu_0 \epsilon_0 \frac{d}{dt} \iint \vec{E} \cdot d\vec{a} \Leftrightarrow B2\pi r = \mu_0 \epsilon_0 \frac{d}{dt} \frac{It}{\epsilon_0 \pi R^2} \pi r^2$$

$$\vec{B} = \frac{I\mu_0}{2\pi R^2} r \hat{\phi}$$

Poynting vector:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \frac{It}{\epsilon_0 \pi R^2} \hat{z} \times \frac{I\mu_0}{2\pi R^2} r \hat{\phi} \bigg|_{r=R} \hat{\phi} = -\frac{I^2 t}{2\epsilon_0 \pi^2 R^3} \hat{r}$$

Power flux:

$$P = \frac{I^2 t}{2\epsilon_0 \pi^2 R^3} 2\pi R d = \frac{I^2 t d}{\epsilon_0 \pi R^2}$$

Total energy in capacitor:

$$U = \frac{1}{2} \epsilon_0 \int E^2 d\tau = \frac{1}{2} \epsilon_0 E^2 \pi R^2 d = \frac{I^2 t^2}{\epsilon_0 \pi R^2} \frac{1}{2} d$$

When taking the derivative of this will give the power flux.

Concept Question: Inductor

The figures above show a side and top view of a solenoid carrying current I with electric and magnetic fields E and B at time t . In the solenoid, the current I is:

1. Increasing in time
2. Constant in time.
3. Decreasing in time.

Problem: Inductor

A solenoid of radius a and length h has an increasing current $I(t)$ as pictured. Consider a point P at radius r ($r < a$). We assume that $I(t)$ is linear proportional to the time, i.e. $\sim \alpha t$

1. Find the magnetic field $B(t)$ at P vs. time t
2. Find the electric field $E(t)$ at P
3. Find the Poynting vector $S(t)$ at P
4. What is the total power flux into/out of the inductor?
5. Does this make sense? How? (Hint: What's U?)

3. Decreasing in time as on the surface of the inductor is pointing outward; induced electric field is counter clockwise so emf is pointing up in the same direction as B , so B is decreasing (Lenz' law).
4. Use ampere's law to determine the magnetic field in the inductor (note that because we assume that the current is linear proportional in time that the E is constant and Maxwell's corrections in Ampere's law is zero):

$$\oint \vec{B} \cdot d\vec{s} = \mu_o \iint \vec{J} \cdot d\vec{a} \Leftrightarrow Bl = \mu_o n l I \Leftrightarrow \vec{B} = \mu_o n I \hat{z}$$

Use Faraday's law to find E at P:

$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{a} \Leftrightarrow \vec{E} 2\pi r = -\mu_o n \pi r^2 \frac{dI}{dt} \hat{\phi} \Leftrightarrow \vec{E} = -\frac{\mu_o n r}{2} \frac{dI}{dt} \hat{\phi}$$

Calculate the pointing vector:

$$\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B} = -\frac{1}{\mu_o} \frac{\mu_o n r}{2} \frac{dI}{dt} \hat{\phi} \times \mu_o n I \hat{z} = -\mu_o \frac{n^2 r I}{2} \frac{dI}{dt} \hat{r}$$

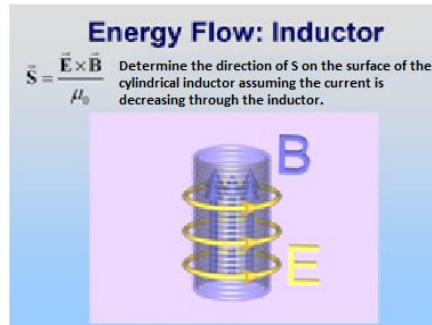
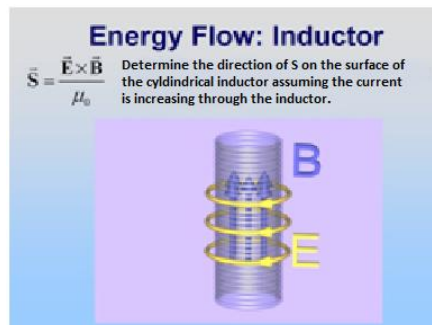
Integrate over surface of device to find energy flux:

$$P = \iint \vec{S} \cdot d\vec{a} = \mu_o \frac{n^2 R I}{2} \frac{dI}{dt} 2\pi R h = \mu_o n^2 R^2 I \pi h \frac{dI}{dt}$$

Total energy stored in inductor integrate energy density over volume of inductor:

$$U = \iiint \frac{B^2}{2\mu_o} d\tau = \frac{B^2}{2\mu_o} \pi R^2 h = \frac{\mu_o n^2 I^2 \pi R^2 h}{2}$$

Taking the derivative would give P.



5. \vec{S} is pointing into the device, so current should be increasing.
6. \vec{S} is pointing out the device, so current should be decreasing.

Using the Poynting vector demonstrate energy transport in a circuit that contains a function-generator (sine wave) connected to an inductor. Assume that the function generator provides a sinusoidal current to the inductor. You might need to review the I-V relation of an inductor. Use the paper of Gallili and Gohlbarg and draw figures similar to fig.3 for different moment in times.

P7. Note that the current and voltage across an inductor are not in phase with each other. Faraday's law tells us that:

$$emf = \oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{a} = -\frac{\partial \Phi_B}{\partial t}$$

Where the left side is the induced emf. Note that the magnetic field of a solenoid can be obtained from Ampere's law and is proportional to the current, or

$$emf = -\frac{dLI}{dt} = -L \frac{dI}{dt}$$

When using the sign convention common in engineering (i.e. direction of the current flows from the high electric potential lead to the low electric potential lead and replacing the emf by V we get:

$$V = L \frac{dI}{dt} \quad [3]$$

This is a well-known result and shows that the current through an inductor and the voltage across it are related via a derivative. This equation leads to an interesting I-V relation for an inductor. If we assume that the inductor is connected to a function generator that sources a sinusoidal current $I = I_o \sin(\omega t)$ through the inductor (see Fig. 1a) then the voltage across the inductor can be calculated from equation [3], and is equal to:

$$V = L \frac{d}{dt} [I_o \sin(\omega t)] = LI_o \omega \cos(\omega t) \quad [4]$$

So current and voltage are no longer in phase, but phase shifted. The current lags 90 degrees behind the voltage. The current voltage relation for an inductor assuming the current is a sine function is given in the graph shown in the figure below.

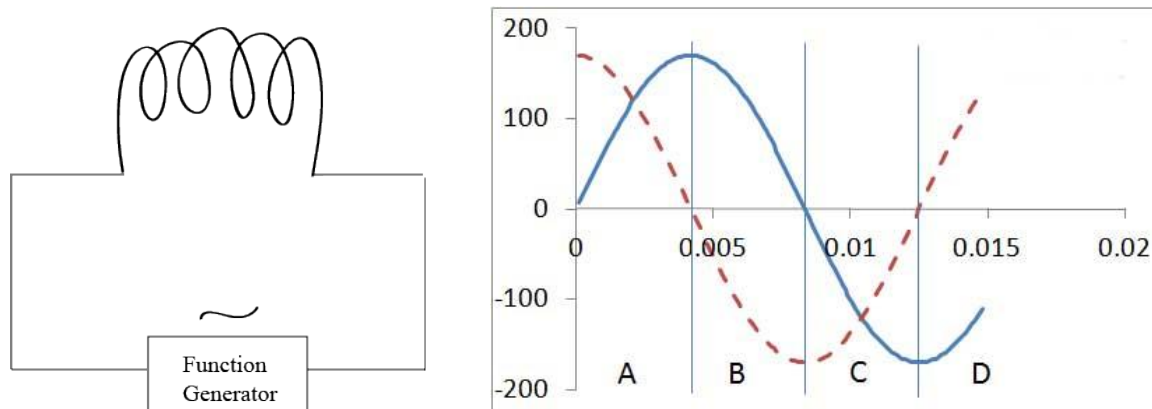


Fig. 1: (a) Inductor connected to function generator; (b) voltage (red) across and current (blue) through an inductor.

The energy flow in the circuit is given by the following relation:

$$P = IV$$

Note that the sign of the power flips and that during the period A the power is positive since both current and electric are potential and energy is transported from the function generator to the inductor. During the 2nd interval (B) the voltage is negative and the current still positive, so the power is negative and energy is transported from the inductor back into the function generator. During the third interval C, the power is again positive and energy is poured back into the inductor etc.

This flow of energy can also be understood with the Poynting vector approach. Note however that now we have to realize that the electric potential changes in the circuit and also the charge on inductor leads. The figure below shows the current and the wire charge for all four periods.

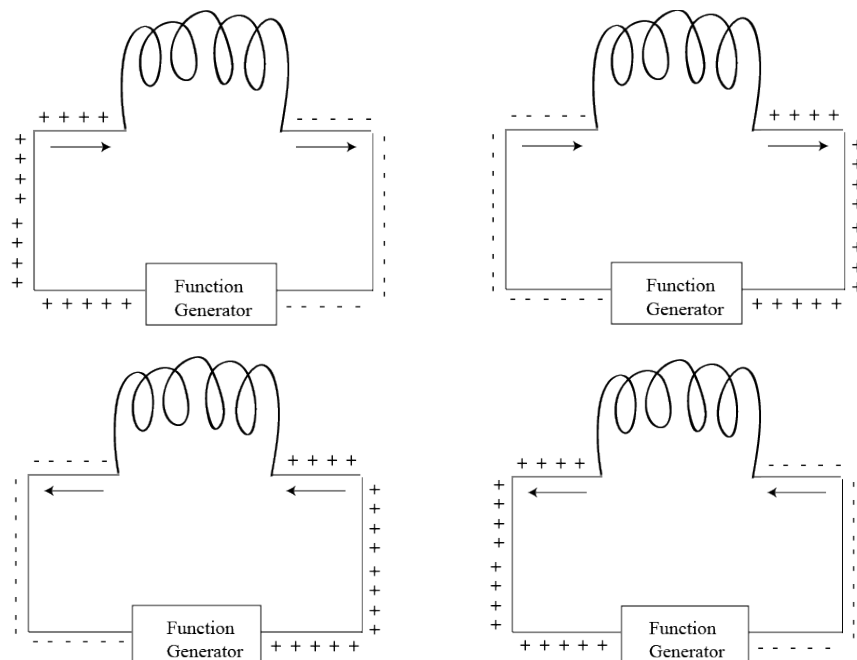


Fig. 2: surface charge and circuit current for interval A, B, C, and D (see Fig. 1 for definition of intervals).

We can use the right hand rule and our knowledge of electrostatics to include the direction of the E and B vectors at different positions along the inductor's leads. Furthermore we can use the information Maxwell's equations to include the E and the B vector in the inductor:

$$\nabla \times \vec{B} = \mu_o \vec{J}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Note that for interval A, the current is increasing, so the flux is increasing, so the curl of E should be opposite to B, etc. etc. The figure below shows the E-field (red) and the B-fields (green).

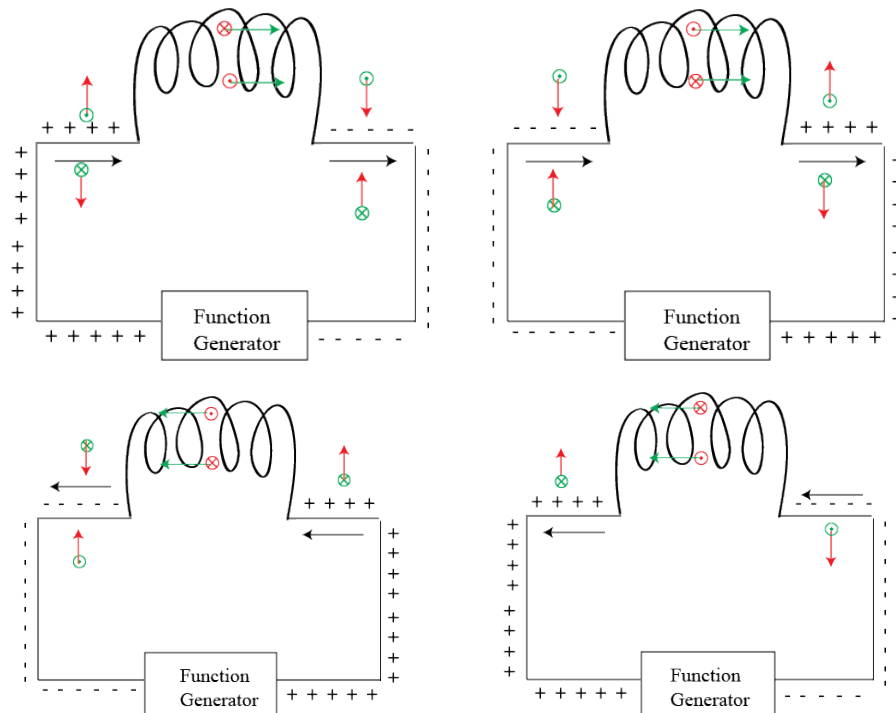


Fig. 3: B-field (green) and E-field at different positions in the circuit for interval A, B, C, and D.

Now it is easy to complete the exercise by constructing the S vector from E and B according to:

$$\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B}$$

The results of this exercise is shown in Fig. 4 below. Note that the energy flow obtained from S for the various intervals is similar to the energy flow obtained from the sign of P.

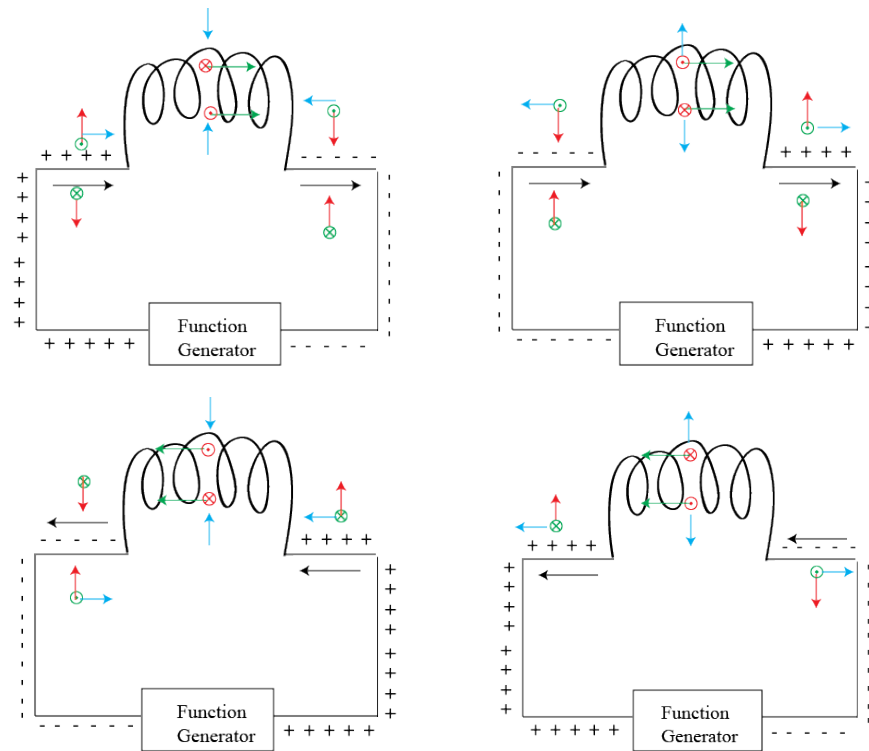


Fig. 4: S vector (blue) incorporated in the figure for interval A, B, C, and D.

Adapted from MIT OPEN COURSEWARE: <http://ocw.mit.edu/courses/physics/8-02sc-physics-ii-electricity-and-magnetism-fall-2010/electromagnetic-waves/energy-and-momentum-in-electromagnetic-waves/>