

Summary of equations chapters 8.

In chapter 8 we discussed three different conservations laws, i.e. charge, energy, and momentum. Conservation law of charge is implied in the continuity equation which is implied in Maxwell's equations.

Energy conservation is still valid for mechanical-EMT problems if one assumes energy to be stored in the electric and magnetic fields, i.e.

$$u_{em} = \frac{1}{2} \left(\epsilon_o E^2 + \frac{B^2}{\mu_o} \right) \quad [1]$$

In class we derived an expression for the amount of work per unit time on a EM system. We started of by calculating the amount of work $\mathbf{F} \cdot d\mathbf{l}$ on a single charge in an electric field \mathbf{E} and a magnetic field \mathbf{B} :

$$\vec{F} \cdot d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} dt = q\vec{E} \cdot \vec{v} dt \quad [2]$$

For the 2nd step of equation [2] we see that we lose the 2nd term: note that $\mathbf{v} \times \mathbf{B} \cdot \mathbf{v}$ should be zero since it is the dot product of a vector perpendicular to \mathbf{v} and \mathbf{v} itself (remember magnetic fields do not do work). For our whole EM system we need to take the integral over the volume. Note that $q = \rho d\tau$ and $\rho \mathbf{v} = \mathbf{J}$ as some of you already noticed in class:

$$\frac{dW}{dt} = \iiint_V (\vec{E} \cdot \vec{J}) d\tau \quad [3]$$

We converted this equation by using Ampere's-Maxwell's law, the 6th product rule on page 21, and some standard Calculus into:

$$\frac{dW}{dt} = -\frac{d}{dt} \iiint_V \frac{1}{2} \left(\epsilon_o E^2 + \frac{1}{2\mu_o B^2} \right) d\tau - \frac{1}{\mu_o} \oint_S (\vec{E} \times \vec{B}) \cdot d\vec{a} \quad [4]$$

So the work done on the charges and currents by the fields per unit time will be equal to a decrease of energy stored in the field (first term on the right) and the rate with which energy is transported out of V (2nd term of the right). Note that the 2nd integral is equal to zero if the magnetic field is equal to zero as the magnetic fields are produced by moving charges and no magnetic field means no 2nd term: so only charge distribution changes (i.e. currents) in V can lead to a non-zero 2nd integral, i.e. non-zero energy radiation term. This equation is often referred to as Poynting's theorem, i.e. the work energy theorem of electrodynamics.

Although Griffith does not address where this energy transporting through S out of V goes to, we can get some understanding from his derivation of equation 8.10 on page 357 and 358. Note that Griffith starts of with assuming that all charges are in V at the bottom of page 357. Note that V is not all of space to infinity and beyond, so part of space is outside V . Energy that is radiated out of V has to be stored in the fields outside of V since there is no charges outside of V on which the fields can do work. So the 2nd integral in equation [4] can only be non-zero if the fields outside V vary as a function of time. So for the particular case where the field outside V are stationary, the 2nd integral has to be zero. If the 2nd integral

would be non-zero where would the radiated energy go? There is no charges outside of V on which the fields can do work and the fields outside V are stationary so the total energy stored in the field outside V is constant. So for stationary cases although S can be non-zero zero at the surface of V , the integral of S over the surface has to be zero. Let us discuss two stationary configuration to get a better understanding of this:

1. A point charge and magnetic dipole at the origin: Let us assume my volume V is a sphere around the origin. The \mathbf{E} -field and \mathbf{B} -field of this particular configuration is given in the figure 1 a below.

Note that the pointing vector S , i.e. $\frac{1}{\mu_0}(\vec{E} \times \vec{B})$ is non-zero over the surface of the sphere. Since

\mathbf{E} is everywhere perpendicular to the sphere and \mathbf{S} is perpendicular to \mathbf{E} and \mathbf{B} , \mathbf{S} is everywhere tangential to the surface of the sphere, so I do not expect a component of S normal to the sphere and so there is no energy flux going through my spherical surface, so the 2nd integral of equation [4] is indeed equal to zero.

2. A point charge at $(0, y_0, 0)$ and a magnetic dipole at $(0, -y_0, 0)$: Let us assume my volume V is again a sphere oriented at the origin with radius larger than y_0 so all charges and magnetic dipoles are in volume V . See Fig. 1b below. Note that for all points on the blue circle, the S is tangential to the sphere perpendicular to the plane of the paper. However for points on the sphere that are not in the plane of the paper, the S will have a component perpendicular to the spherical surface and a not zero flux density. Consider for example a point on the equator that intersects with the positive x -axis: \mathbf{B} will be in the negative z -direction and \mathbf{E} will have a component in the negative y -direction and positive x -direction. As \mathbf{S} is proportional to \mathbf{E} cross \mathbf{B} , \mathbf{S} should have a component in the x and y -direction and thus a normal component on the surface of the sphere. A top view is sketched in Fig. 1c. For this particular charge-magnetic dipole configuration, the perpendicular component of \mathbf{S} is not zero for all points on the sphere. For every area on the sphere however where the flux is positive a similar area can be found where the flux is negative. So I can use symmetry to convince myself that integrating \mathbf{S} over the surface of the sphere will result in a net zero flux through the sphere's surface (see also Fig. 1 b below). For the areas of positive S energy is transferred out of V and for areas with a negative S energy is transferred back into V : so no net energy transport across the surface of the sphere, as should be from equation [4].

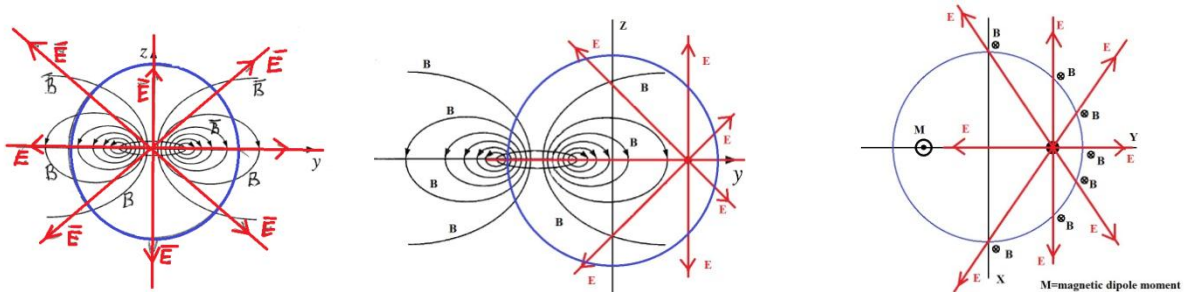


Fig. 1: (a) \mathbf{B} and \mathbf{E} for a charge and magnetic dipole both positioned at the origin; (b) \mathbf{E} and \mathbf{B} for a magnetic dipole at $(0, -y_0, 0)$ and an electric dipole at $(0, y_0, 0)$ side view; (c) same as (b) but now top view.

The cross product under the integral on the right side of equation [4] is the energy flux density and identifies the amount of energy that flows across the boundary S per second. Read increase in mechanical energy per unit time in volume (dW/dt) is equal to decrease of EM energy in V and transfer of EM energy out of V through S .

In the particular case that there is no work done on the charges the left side of equation [4] is zero. So now we can rewrite equation [4] as:

$$\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\nabla \cdot \vec{S} = -\nabla \cdot \frac{1}{\mu_o}(\vec{E} \times \vec{B}) \quad [5]$$

This is sometimes referred to as EM-energy conservation equation and looks similar to the continuity equation. In general though dW/dt will be unequal to zero. For problem 8.2 one can apply equation [5].

Section 8.2.1 starts with showing that Newton's third law might not be obeyed for the interaction between two moving positive charges. In order to still be able to use momentum conservation laws in electrodynamics, one therefore associates a certain momentum with an electromagnetic field. The derivation of this expression is complicated and involves the Maxwell stress tensor which is beyond the scope of our undergraduate EMT2 class. So we will skip section 8.2.2 but use the equations in section 8.2.3. For the change of the mechanical momentum of the particles in volume V we find the following expression:

$$\frac{\partial}{\partial t} \vec{p}_{mech} = -\epsilon_o \mu_o \frac{d}{dt} \iiint_V \vec{S} d\tau + \iint_S \vec{T} \cdot d\vec{a} \quad [5]$$

Where \vec{p} is the linear momentum, and \vec{T} is the stress tensor whose negative value is the momentum flux density (defines the flow of momentum). The quantity $\epsilon_o \mu_o \vec{S}$ is the momentum density of the field, \vec{g} :

$$\vec{g} = \epsilon_o \mu_o \vec{S} = \epsilon_o \vec{E} \times \vec{B} \quad [6]$$

We did not really worry about the definition of the stress tensor and leave that to graduate EMT. In order to find the total linear momentum of a certain charge and current distribution we have to integrate $\epsilon_o \mu_o \vec{S}$ over the volume for which S is unequal to zero. The linear momentum stored in the field can be transferred into mechanical linear momentum by changing the charge distribution so that the integral of $\epsilon_o \mu_o \vec{S}$ over the volume goes to zero. Note that if for each point in space \vec{E} or \vec{B} is zero, no momentum is stored in the Electromagnetic field. If in certain areas of space \vec{E} and \vec{B} are both non-zero a net momentum can be stored in the electromagnetic field. One can get this momentum out of the field by reconfiguring the position of the charges and considering the impulse associated with this reconfiguration. Although there are two types of forces, i.e. electrostatic forces and magnetostatic forces (Lorentz force), only the latter needs to be considered when calculating the net impulse on your system when rearranging the charge distribution. Note that Newton's third law is still valid for

electrostatics. This means that when we rearrange the charges to lower the momentum stored in the field, the impulse acting on one part of the system because of the electrostatic forces is equal and opposite to the impulse working on another part of the system, so there is basically no net impulse of electrostatic origin. As Newton's third law breaks down for the magnetic forces, when rearranging the charges a net impulse is working on the system originating from the Lorentz force. This non-zero impulse will change the mechanical momentum of your system. The association of momentum density with a field as given by equation [6] above, is a fix for Newton's third law and the momentum conservation law for systems that include magnetic fields.

We studied examples where momentum is transferred from the fields to the mechanical domain: for example discharging a capacitor with a solid wire connected to both charged plates (example 8.3, problem 8.5, and problem 8.6).

We skipped 8.2.4 and 8.3.