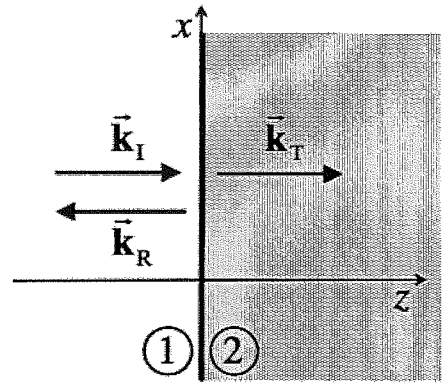
12A - R & T (Normal Incidence) NAME_____

A. An electromagnetic plane wave is traveling through region **1** (a material with index of refraction n_1), and encounters a material in region **2** (index of refraction n_2), with the boundary located at $z = 0$. The incident wave (I) gives rise to a reflected wave (R) and a transmitted wave (T).



With the incident wave linearly polarized in the x-direction, the electric fields of the incident, reflected and transmitted waves can each be represented by the following complex exponentials:

$$\vec{\mathbf{E}}_I(\vec{r},t) = E_I \exp\left[i\left(k_I z - \omega_I t\right)\right]\hat{x}$$

$$\vec{\mathbf{E}}_R(\vec{r},t) = E_R \exp\left[i\left(-k_R z - \omega_R t + \delta_R\right)\right]\hat{x}$$

$$\vec{\mathbf{E}}_T(\vec{r},t) = E_T \exp\left[i\left(k_T z - \omega_T t\right)\right]\hat{x}$$

According to Faraday's Law, the parallel component of the total electric field on either side of the boundary must be the same at all times:

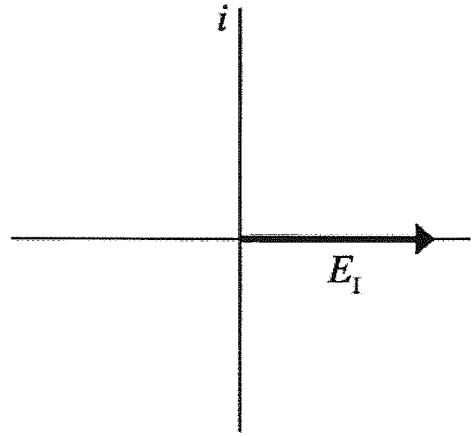
$$\vec{\mathbf{E}}_1^{\parallel}(z=0,t) = \vec{\mathbf{E}}_2^{\parallel}(z=0,t) \quad \rightarrow \quad \vec{\mathbf{E}}_I^{\parallel}(z=0,t) + \vec{\mathbf{E}}_R^{\parallel}(z=0,t) = \vec{\mathbf{E}}_T^{\parallel}(z=0,t)$$

Use the information given to re-write this boundary condition using the complex exponential notation from above.

Now, write out this boundary condition at $t = 0$.

▼ You may continue, but be sure to check your answers with an instructor.

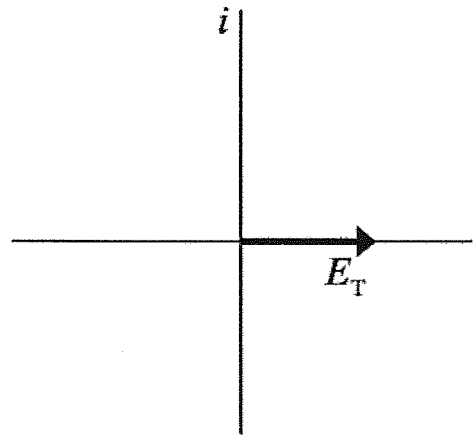
B. The arrows in the two diagrams at right represent the electric fields of the incident and transmitted waves in the complex plane at $t = 0$. Notice that in this case $E_{\mathrm{I}} > E_{\mathrm{T}}$



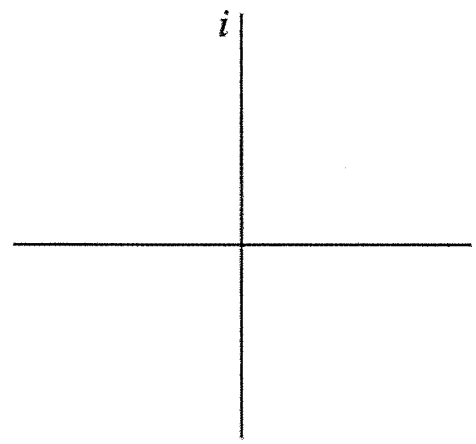
In order for the following boundary condition to be satisfied

$$E_{\mathrm{I}} + E_{\mathrm{R}} \exp(i\delta_R) = E_{\mathrm{T}}$$

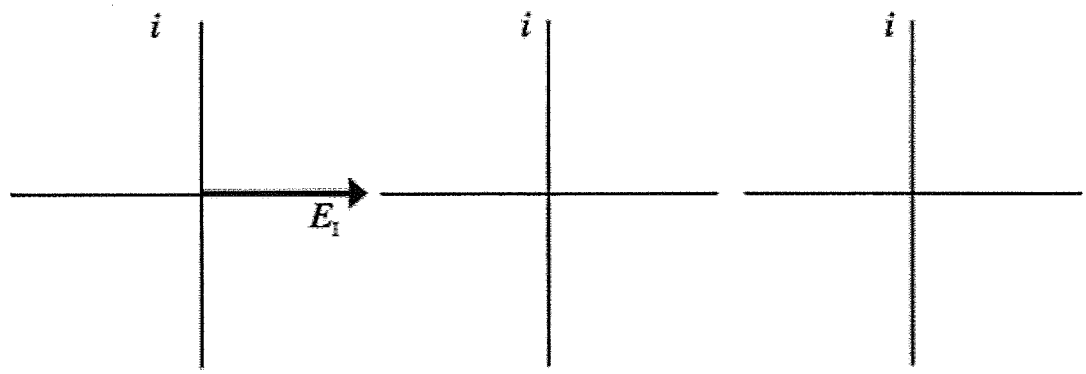
what is the phase shift δ_{R} for the *reflected* wave? [Hint: $\exp(i\pi) = -1$]



Use the axes at right to draw an arrow representing the magnitude and direction of the electric field for the *reflected* wave at $t = 0$.



C. With the *incident* E-field at $t = 0$ shown below at left, use the remaining axes to draw arrows representing the E-fields of the *reflected* and *transmitted* waves for the case $E_T > E_I$. Be sure to label your arrows.



What is the phase shift δ_R of the *reflected* wave in this case?

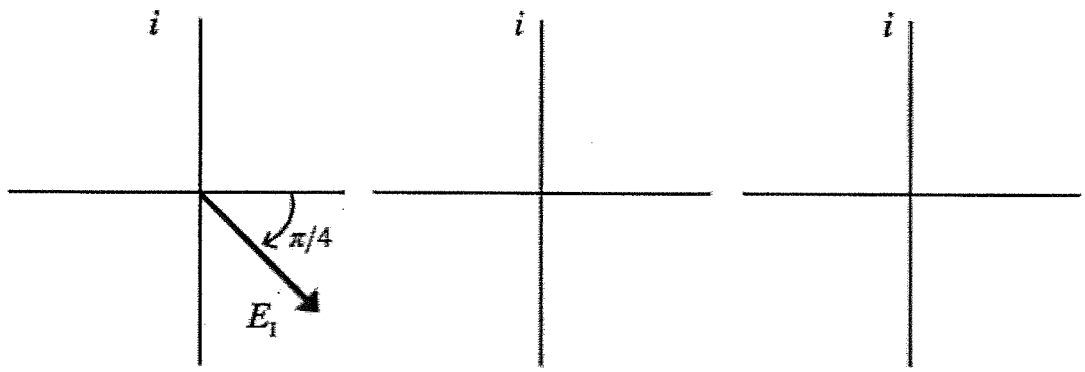
Use the diagram above to draw the *incident* E-field at a later time $\omega_I t = \pi/4$. Be sure to think carefully about which direction $E_I \exp(-i\omega_I t)$ rotates in the complex plane.

Boundary conditions must be satisfied at all times (not just at $t = 0$). If we solve for E_I in the time-dependent equation from the first page:

$$E_I = \pm E_R \exp\left[-i(\omega_R - \omega_I)t\right] + E_T \exp\left[-i(\omega_T - \omega_I)t\right]$$

Notice that the quantity on the *left side* of the equation is an entirely real constant. For this equality to hold at all times, what must be true about the quantities $(\omega_R - \omega_I)$ and $(\omega_T - \omega_I)$? Briefly explain your reasoning.

D. With the *incident* E-field at $\omega_1 t = \pi/4$ shown below at left, use the remaining axes to draw arrows representing the E-fields of the *reflected* and *transmitted* waves for the case $E_T > E_I$. Be sure to label your arrows.



E. Assume that both materials have a negligible magnetic permeability ($\mu_1 = \mu_2 = \mu_0$). The following four boundary conditions for the electric and magnetic fields must be true at the interface of materials 1 & 2:

$$i)\ \varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp} \qquad ii)\ B_1^{\perp} = B_2^{\perp}$$

$$iii)\ \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel} \qquad iv)\ \mathbf{B}_1^{\parallel} = \mathbf{B}_2^{\parallel}$$

Which of these four boundary conditions (i-iv) could be used to get the following equation? [v_1 & v_2 are the wave speeds in regions 1 & 2]

$$E_I - E_R e^{i\delta_R} = \frac{v_1}{v_2} E_T$$

[Hint: Recall that for EM plane waves $\vec{k} \times \vec{\mathbf{E}} = \omega \vec{\mathbf{B}}$.]

Show how you can arrive at this boundary equation by taking the direction of propagation for the reflected wave into consideration.

F. For simplicity, let $v_1/v_2 = n_2/n_1 \equiv \beta$ in these two boundary equations, where E_R could be positive or negative, depending on the phase shift:

(Eq. I) $E_I + E_R = E_T$ **(Eq. II)** $E_I - E_R = \beta E_T$

Use equations **I** & **II** to solve for E_T in terms of E_I. Explain in words whether your answer makes sense in the limit $n_2 \to n_1$.

Use this result to solve for E_R in terms of E_I. Explain in words whether your answer makes sense in the limit $n_2 \to n_1$.

For what values of $\beta = n_2/n_1$ will the *reflected* wave be 180^0 out of phase with the incident wave?

If the *reflected* wave is 180^0 out of phase with the incident wave, does this mean that light in medium 2 travels *faster* or *slower* than light in medium 1?