HW8_2

1. Work problem 8.2.

Problem 8.2

(a) $\mathbf{E} = \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}};\ \sigma = \dfrac{Q}{\pi a^2};\ Q(t) = It \ \Rightarrow\ \mathbf{E}(t) = \boxed{\dfrac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}}.$

$$B\,2\pi s = \mu_0\epsilon_0 \frac{\partial E}{\partial t}\pi s^2 = \mu_0\epsilon_0 \frac{I\pi s^2}{\pi\epsilon_0 a^2} \ \Rightarrow\ \mathbf{B}(s,t) = \boxed{\frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}.}$$

(b) $u_{\text{em}} = \dfrac{1}{2}\left(\epsilon_0 E^2 + \dfrac{1}{\mu_0}B^2\right) = \dfrac{1}{2}\left[\epsilon_0\left(\dfrac{It}{\pi\epsilon_0 a^2}\right)^2 + \dfrac{1}{\mu_0}\left(\dfrac{\mu_0 I s}{2\pi a^2}\right)^2\right] = \boxed{\dfrac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right].}$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{It}{\pi\epsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{\mathbf{s}}) = \boxed{-\frac{I^2 t}{2\pi^2\epsilon_0 a^4}\,s\,\hat{\mathbf{s}}.}$$

$$\frac{\partial u_{\text{em}}}{\partial t} = \frac{\mu_0 I^2}{2\pi^2 a^4}2c^2 t = \frac{I^2 t}{\pi^2\epsilon_0 a^4};\quad -\boldsymbol{\nabla}\cdot\mathbf{S} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\boldsymbol{\nabla}\cdot(s\,\hat{\mathbf{s}}) = \frac{I^2 t}{\pi^2\epsilon_0 a^2} = \frac{\partial u_{\text{em}}}{\partial t}.\ \checkmark$$

(c) $U_{\text{em}} = \displaystyle\int u_{\text{em}} w 2\pi s\,ds = 2\pi w \frac{\mu_0 I^2}{2\pi^2 a^4}\int_0^b [(ct)^2 + (s/2)^2] s\,ds = \frac{\mu_0 w I^2}{\pi a^4}\left[(ct)^2\frac{s^2}{2} + \frac{1}{4}\frac{s^4}{4}\right]\Bigg|_0^b$

$= \boxed{\dfrac{\mu_0 w I^2 b^2}{2\pi a^4}\left[(ct)^2 + \dfrac{b^2}{16}\right].}$ Over a surface at radius b: $P_{\text{in}} = -\displaystyle\int \mathbf{S}\cdot d\mathbf{a} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\left[b\,\hat{\mathbf{s}}\cdot(2\pi b w\,\hat{\mathbf{s}})\right] = \boxed{\frac{I^2 w t b^2}{\pi\epsilon_0 a^4}}.$

$\dfrac{dU_{\text{em}}}{dt} = \dfrac{\mu_0 w I^2 b^2}{2\pi a^4}2c^2 t = \dfrac{I^2 w t b^2}{\pi\epsilon_0 a^4} = P_{\text{in}}.\ \checkmark$ (Set $b = a$ for *total*.)

2. Work problem 8.5. Problem is sketched in the figures below. Note that surface charge density times velocity means surface current density:

$$\vec{K} = \sigma\,\vec{v}$$

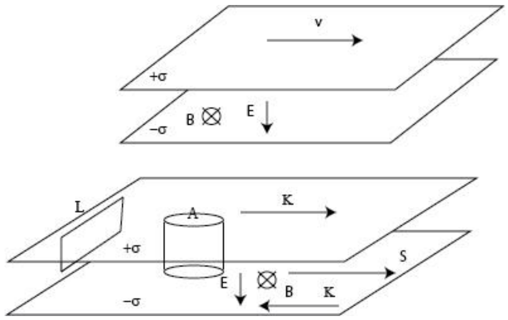
E goes from top plate to bottom plate (positive to negative charge) and is zero outside the plates. B in between the plates goes into the plane of the picture and is zero outside the plates.

I calculate E from Gauss' law. I use the fact that superposition tells me there is no field above the top plate and no field below the bottom plate. Furthermore I know for preceding chapters that the electric field in between the plates is constant. First I choose a proper Gaussian surface, in this case a pillbox with cap surface A. The top cap is placed above the top plate and the bottom cap is placed in between the plates. Applying Gauss' law gives me:

$$\oiint_{can} \vec{E}\bullet d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_o} \Rightarrow EA = \frac{\sigma A}{\varepsilon_o} \Rightarrow E = \frac{\sigma}{\varepsilon_o}$$

I calculate B in between the plates from Ampere's law. I noticed that E is constant so I do not need Maxwell's extension on Ampere's law. Furthermore I know from previous chapters that the magnetic field is independent of the distance to the plates. Superposition tells me now that there is no magnetic field above the top plate and no magnetic field below the bottom plate:

$$\oint_{Amperian\,loop} \vec{B} \bullet d\vec{l} = \mu_o I_{enclosed} \Rightarrow BL = \mu_o KL \Rightarrow B = \mu_o K$$



For the momentum density I use:

$$\vec{g} = \varepsilon_o \vec{E} \times \vec{B} = \mu_o \vec{K} \sigma$$

So total momentum stored in the field per surface area is volume time momentum density, i.e.

$$gAd = \mu_o \sigma v \sigma Ad = \mu_o v \sigma^2 Ad$$

b. Only consider the impulse originating from the Lorentz force. See also summary of chapter 8 and remember our discussion about momentum conservation when two charged particles approach each other under 90 degrees. Newton's third law was obeyed for the electric field and only broke down for the Lorentz force. Of course momentum conservation and Newton's third law are strongly related to each other.

So the Lorentz force is:

$$F = quB = \sigma Au \mu_o K = \sigma Au \mu_o \sigma v$$

Impulse is now the force integrated over time or:

$$\mathrm{Im}\,pulse = \int F dt = \mu_o \sigma^2 v Au \frac{d}{u} = \mu_o \sigma^2 v Ad$$

3. Work problem 8.6.
   a. See figure in the book for configuration. Momentum density is:

$$g = \varepsilon_o \left(\vec{E} \times \vec{B}\right) = \varepsilon_o \mu_o S$$

So total momentum is density time volume as the momentum density is constant between the plates this will be:

$$p = Ad\varepsilon_o EB$$

   b. From a previous chapter I learned that the discharge current is:

$$I(t) = \frac{V_o}{R}\left(e^{-t/(RC)} - 1\right)$$

Where V_o is the initial voltage across the capacitor. This can be related to the initial electric field across the capacitor by:

$$E = V / d$$

The force on the resistive wire (note that this little wire is placed in a magnetic field B and that current and field are perpendicular to each other:

$$F = IdB = \frac{Ed}{R} d\left(e^{-t/(CR)} - 1\right)B$$

The impulse will be the integral with respect to time from 0 to infinity, i.e.

$$\int_0^\infty F dt = \int_0^\infty \frac{Ed^2}{R} B\left(e^{-t/(RC)} - 1\right)dt = \frac{Ed^2}{R} B\left(-RCe^{-t/(RC)}\right)\Big|_0^\infty = Ed^2 BC = EdB\varepsilon_o A$$