Key HW 9_1

1. Work problem 9.1

Problem 9.1

$$\frac{\partial f_1}{\partial z} = -2Ab(z-vt)e^{-b(z-vt)^2};\ \frac{\partial^2 f_1}{\partial z^2} = -2Ab\left[e^{-b(z-vt)^2} - 2b(z-vt)^2 e^{-b(z-vt)^2}\right];$$

$$\frac{\partial f_1}{\partial t} = 2Abv(z-vt)e^{-b(z-vt)^2};\ \frac{\partial^2 f_1}{\partial t^2} = 2Abv\left[-ve^{-b(z-vt)^2} + 2bv(z-vt)^2 e^{-b(z-vt)^2}\right] = v^2\frac{\partial^2 f_1}{\partial z^2}.\ \checkmark$$

$$\frac{\partial f_2}{\partial z} = Ab\cos[b(z-vt)];\ \frac{\partial^2 f_2}{\partial z^2} = -Ab^2\sin[b(z-vt)];$$

$$\frac{\partial f_2}{\partial t} = -Abv\cos[b(z-vt)];\ \frac{\partial^2 f_2}{\partial t^2} = -Ab^2v^2\sin[b(z-vt)] = v^2\frac{\partial^2 f_2}{\partial z^2}.\ \checkmark$$

$$\frac{\partial f_3}{\partial z} = \frac{-2Ab(z-vt)}{[b(z-vt)^2+1]^2};\ \frac{\partial^2 f_3}{\partial z^2} = \frac{-2Ab}{[b(z-vt)^2+1]^2} + \frac{8Ab^2(z-vt)^2}{[b(z-vt)^2+1]^3};$$

$$\frac{\partial f_3}{\partial t} = \frac{2Abv(z-vt)}{[b(z-vt)^2+1]^2};\ \frac{\partial^2 f_3}{\partial t^2} = \frac{-2Abv^2}{[b(z-vt)^2+1]^2} + \frac{8Ab^2v^2(z-vt)^2}{[b(z-vt)^2+1]^3} = v^2\frac{\partial^2 f_3}{\partial z^2}.\ \checkmark$$

$$\frac{\partial f_4}{\partial z} = -2Ab^2ze^{-b(bz^2+vt)};\ \frac{\partial^2 f_4}{\partial z^2} = -2Ab^2\left[e^{-b(bz^2+vt)} - 2b^2z^2e^{-b(bz^2+vt)}\right];$$

$$\frac{\partial f_4}{\partial t} = -Abve^{-b(bz^2+vt)};\ \frac{\partial^2 f_4}{\partial t^2} = Ab^2v^2e^{-b(bz^2+vt)} \neq v^2\frac{\partial^2 f_4}{\partial z^2}.$$

$$\frac{\partial f_5}{\partial z} = Ab\cos(bz)\cos(bvt)^3;\ \frac{\partial^2 f_5}{\partial z^2} = -Ab^2\sin(bz)\cos(bvt)^3;\ \frac{\partial f_5}{\partial t} = -3Ab^3v^3t^2\sin(bz)\sin(bvt)^3;$$

$$\frac{\partial^2 f_5}{\partial t^2} = -6Ab^3v^3t\sin(bz)\sin(bvt)^3 - 9Ab^6v^6t^4\sin(bz)\cos(bvt)^3 \neq v^2\frac{\partial^2 f_5}{\partial z^2}.$$

2. Work problem 9.4

Problem 9.4

The wave equation (Eq. 9.2) says $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$. Look for solutions of the form $f(z,t) = Z(z)T(t)$. Plug this in: $T\frac{d^2Z}{dz^2} = \frac{1}{v^2}Z\frac{d^2T}{dt^2}$. Divide by ZT : $\frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{v^2 T}\frac{d^2T}{dt^2}$. The left side depends only on z, and the right side only on t, so both must be constant. Call the constant $-k^2$.

$$\left\{\begin{array}{lcll} \frac{d^2Z}{dz^2} & = & -k^2 Z & \Rightarrow \quad Z(z) = Ae^{ikz} + Be^{-ikz}, \\ \frac{d^2T}{dt^2} & = & -(kv)^2 T & \Rightarrow \quad T(t) = Ce^{ikvt} + De^{-ikvt}. \end{array}\right\}$$

(Note that k must be *real*, else Z and T blow up; with no loss of generality we can assume k is *positive*.) $f(z,t) = \left(Ae^{ikz} + Be^{-ikz}\right)\left(Ce^{ikvt} + De^{-ikvt}\right) = A_1 e^{i(kz+kvt)} + A_2 e^{i(kz-kvt)} + A_3 e^{i(-kz+kvt)} + A_4 e^{i(-kz-kvt)}$.
The general linear combination of separable solutions is therefore

$$f(z,t) = \int_0^\infty \left[A_1(k)e^{i(kz+\omega t)} + A_2(k)e^{i(kz-\omega t)} + A_3(k)e^{i(-kz+\omega t)} + A_4(k)e^{i(-kz-\omega t)}\right] dk,$$

where $\omega \equiv kv$. But we can combine the third term with the first, by allowing k to run *negative* ($\omega = |k|v$ remains positive); likewise the second and the fourth:

$$f(z,t) = \int_{-\infty}^{\infty} \left[A_1(k)e^{i(kz+\omega t)} + A_2(k)e^{i(kz-\omega t)}\right] dk.$$

Because (in the end) we shall only want the the *real part* of f, it suffices to keep only *one* of these terms (since k goes negative, both terms include waves traveling in both directions); the second is traditional (though either would do). Specifically,

$$\mathrm{Re}(f) = \int_{-\infty}^{\infty} [\mathrm{Re}(A_1)\cos(kz+\omega t) - \mathrm{Im}(A_1)\sin(kz+\omega t) + \mathrm{Re}(A_2)\cos(kz-\omega t) - \mathrm{Im}(A_2)\sin(kz-\omega t)]\, dk.$$

The first term, $\cos(kz+\omega t) = \cos(-kz-\omega t)$, combines with the third, $\cos(kz-\omega t)$, since the negative k is picked up in the other half of the range of integration, and the second, $\sin(kz+\omega t) = -\sin(-kz-\omega t)$, combines with the fourth for the same reason. So the general solution, for our purposes, can be written in the form

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\, dk \quad \text{qed (the tildes remind us that we want the real part).}$$