Conceptual questions:

1. Assume a plane wave in vacuum for which the amplitude of the magnetic field is  $B_o$  and the amplitude of the electric field is  $E_o$ . Determine the units of  $E_o/H_o$ . Surprised? ( $\mu_o=4\pi 10^{-7} \text{ N/A}^2$ )

Key: The quickest way is to use the following relations that are valid for vacuum:

$$B_o = \frac{E_o}{c} \qquad \qquad H = \frac{B}{\mu_o}$$

Note that for non-vacuum the units of the quantities do not change. Combining them gives:

$$\frac{E_o}{H_o} = c\mu_o \Rightarrow \sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{\frac{N}{A^2}}{\frac{C^2}{Nm^2}}} = \sqrt{\frac{N^2m^2}{A^2C^2}} = \sqrt{\frac{V^2m^2}{A^2m^2}} = \frac{V}{m} = Ohm$$

Where we used that a N/C is equal to a V/m. Another quicker way is to use the units of E and H, i.e. V/m and A/m. Note that an electric field implies a potential which normally creates a current or an electric displacement, which causes a magnetic field. So E/H should give you already the impression of resistance.

2. A. The picture below shows a plane wave in vacuum and in a metal. Determine the direction of S at point P1, P2, P3, and P4.



B1. Is the direction of S in the direction of k for a plane wave in vacuum?
B2. Is the direction of S always in the direction of k for a plane wave in a metal?
C. The answer to B2 has implications for the energy flow per unit second of an EM-wave in a metal, the momentum density of and EM-wave in a metal, or for both?
<u>Key:</u> For a plane wave in normal space the S vector is parallel the k-vector for all points of the wave, although the magnitude of the S-vector is not constant. In a metal, or any absorbing medium, the S-vector flips direction along the propagation direction of the wave (you can determine the direction of the S-vector from the cross product of E and H). Of course this has implications for both the energy flow per second as the momentum density as they both are linear proportional to S, i.e:

$$\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B}$$
$$\vec{g} = \mu_o \varepsilon_o \vec{S}$$

3. Use one of the Maxwell's equations to explain me why a plane EM-wave in vacuum is transverse and does not have field components along its k-vector.

<u>Key:</u> Note that transverse means that the **E** and **B**-fields of the wave are perpendicular to the **k**-vector. Transverse does not mean that **E** and **B** are perpendicular. Although the latter is the case it is not the transverse property of the wave.

An EM-wave in a homogeneous isotropic material (including vacuum) not because of the wave-equation but because of the fact that the divergence of  $\mathbf{E}$  and the divergence of  $\mathbf{B}$  are zero. We proofed in class that if one considers both the  $\mathbf{B}$ -field and the  $\mathbf{E}$ -field to be plane waves that propagate in the z-direction that the divergence laws for the fields show that the z-component of the fields are zero, i.e for  $\mathbf{E}$ .

$$\widetilde{\vec{E}} = \widetilde{\vec{E}}_{o} e^{i(kz-\omega t)}$$
$$\nabla \bullet \widetilde{\vec{E}}_{o} = \frac{\partial \widetilde{E}_{ox} e^{i(kz-\omega t)}}{\partial x} + \frac{\partial \widetilde{E}_{oy} e^{i(kz-\omega t)}}{\partial y} + \frac{\partial \widetilde{E}_{oz} e^{i(kz-\omega t)}}{\partial z} = 0 + 0 + \frac{\partial \widetilde{E}_{oz} e^{i(kz-\omega t)}}{\partial z} = 0$$

The last equation is only valid if the z-component of the complex field amplitude of the electric field is zero.

4. A solution to the wave-equation is:

$$f(z,t) = A\cos(-kz - \omega t + \delta)$$

Where we assume that k,  $\omega$ , and  $\delta$  are positive constants.

- a. What is the speed of this wave?
- b. Which way is it moving?
- c. If  $\delta$  is small and positive, is this wave delayed or advanced?
- d. What is the frequency of the wave?

<u>Key:</u> The speed of the wave is given by  $\omega/k$  this can be easily seen as follows. Assume t=0. Then the phase of the wave at z=0 is given by  $\delta$ . Now consider 1 second later in time and determine the z for which the phase is still valid (so you traveled on top of the wave). So:

$$-kz - \omega + \delta = \delta \Longrightarrow -kz = \omega \Longrightarrow z = -\frac{\omega}{k}$$

Or in other words in one second I would have changed my z with  $-\omega/k$  which is the velocity. The speed is the absolute value of this number.

It is clear from the velocity calculated above that the wave is moving in the negative z-direction. The d will shift the cosine in the positive z-direction so since the wave is moving in the negative z-direction a positive d implies a delay. The frequency of the wave can be calculated similar as we calculated the speed above but now we have to look in the time domain only. The period of a cosine is  $2\pi$  so:

$$\omega T = 2\pi \Longrightarrow \frac{1}{T} = \frac{\omega}{2\pi} = f$$

Consider a large parallel plate capacitor as shown, charging so that  $Q = Q_0 + \beta t$  on the positively charged plate. Assuming the edges of the capacitor and the wire connections to the plates can be ignored, what is the magnitude of the magnetic field B halfway between the plates, at a radius  $r^2$ 



Key: I first determine what happens conceptually. The current is constant but the charge on the plates is increasing with constant rate. More charge on the plates means that the electric field between the plates is not constant but increasing in this case linearly with time. The time varying electric field will have a magnetic field associated with it because of Maxwell extension to Ampere's law. And now the math: I need to decide about an Amperian loop, and I choose a circle with radius r concentric with the wires, parallel to the plates, in between the plates of the capacitor. Notice that it is not cool to just write down an integral without identifying the Amperian loop! Applying Ampere's law on the loop gives:

$$\oint_{AL} \vec{B} \bullet d\vec{l} = B2\pi r = \mu_o \varepsilon_o \iint_{S} \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a} = \mu_o \varepsilon_o \pi (r)^2 \frac{\partial \vec{E}}{\partial t}$$
<sup>[1]</sup>

The field between the pole pieces of the capacitor can be calculated from the charge on the plates. Since this is a parallel plate capacitor I use:

$$E = \frac{V}{d} = \frac{Q}{dC} = \frac{Q_o + \beta t}{dC} \Longrightarrow \frac{\partial E}{\partial t} = \frac{\beta}{dC} = \frac{\beta}{\varepsilon_o A}$$
[2]

Where A is the total surface area of the capacitor. Plugging equation [2] in [1] gives:

$$B = \frac{\mu_o r}{2} \frac{\beta}{\pi a^2}$$
[3]

So (E). Note that it is not cool to not choose any answer. The question does not ask for an explanation, so I can at least use an educated guess based on units to eliminate some of the choices and narrow down my choice, so use this opportunity!!

6. Consider a very long solenoid of length *L*, radius *r*, and turns per length *n*. The current *I* in the solenoid is linearly ramped from I = 0 to  $I = I_0$  over a period  $t_0$  as shown in the graph.



- a) Integrate the magnetic field energy density to derive a formula for the total field energy stored in the solenoid at times  $t > t_0$ .
- b) Solve for the electric field everywhere at times  $0 < t < t_0$ .

- c) Solve for the Poynting vector (direction and magnitude) at r = R (just inside the walls of the solenoid) as a function of time *t*.
- d) Show that the total field energy/time passing from the walls of the solenoid into its interior, when integrated from t = 0 to  $t = t_0$ , gives the same total energy as you computed in part a).

<u>Key:</u> (a)Note that this is a solenoid and not a toroid. If you forgot the difference please check in the text book. The magnetic field is given by.

$$B = \mu_0 n I$$

You can also calculate this by applying Ampere's law on a solenoid. If you do so make sure to include a drawing and identify the Amperian loop in the drawing.

And the energy in the solenoid is given by

$$U = \frac{1}{2\mu_0} \int B^2 d\tau = \frac{1}{2\mu_0} [\mu_0 n I_0]^2 (\pi r^2 L) = \frac{\mu_0 \pi r^2 n^2 I_0^2 L}{2}$$

(b) During the time the current is changing, the electric field can be found from

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\mu_0 n \frac{dI}{dt} \hat{\boldsymbol{z}}$$

This tells us that the electric field will be circumferential. We can use Stokes' theorem to say

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a}$$

As our closed curve, we choose a circle of radius s. We then have

$$E(2\pi s) = -\mu_0 n \frac{dI}{dt} (\pi s^2)$$

Or,

$$\boldsymbol{E} = -\mu_0 n \frac{I_0}{2t_0} s \boldsymbol{\widehat{\phi}}$$

(c) During the time the current is changing, it is given by

$$I = \frac{I_0}{t_0}$$

The Poynting vector at the surface of the solenoid is then

$$S = \frac{1}{\mu_0} E \times B = \frac{1}{\mu_0} \left[ -\mu_0 n \frac{I_0}{2t_0} R \widehat{\boldsymbol{\phi}} \right] \times \left[ \mu_0 n \frac{I_0}{t_0} t \widehat{\boldsymbol{z}} \right]$$
$$= -\mu_0 n^2 R \frac{I_0^2 t}{2t_0^2} \widehat{\boldsymbol{s}}$$

(d) The energy flux is given by

$$u=\int \boldsymbol{S}\cdot d\boldsymbol{a}$$

Since the Poynting vector is constant on the surface of the solenoid, we can pull it out of the integral, which gives

$$u = \left[\mu_0 n^2 R \frac{I_0^2 t}{2t_0^2}\right] (2\pi RL) = \mu_0 \pi L n^2 R^2 \frac{I_0^2}{t_0^2} t$$

Integrating from t = 0 to  $t = t_0$  gives the same result as in part a).

7. A capacitor with circular plates of radius *R* separated by distance  $d \ll R$  is being charged by a steady current *I*. The plates are sufficiently close that fringe effects can be ignored.



- a) Compute the magnitude of the B-field between the plates at all distances *r* from the center of the plates (r < R & r > R). Plot the magnitude of the B-field vs. *R*.
- b) Compute the Poynting vector  $\vec{S}$  (magnitude and direction) on the *rim* of the capacitor, between the plates, at r = R. (The "rim" is the ribbon of area at r = R between the plates; see the diagram.)
- c) Show that the rate at which the capacitor's stored energy is increasing is equal to the rate at which field energy is entering through the rim. In other words, show that

$$\frac{dU}{dt} = \int_{RIM} \boldsymbol{S} \cdot d\boldsymbol{a}$$

Key: (a)The electric field between the plates is given by

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0} \, \hat{\boldsymbol{z}} = \frac{Q}{4\pi\varepsilon_0 R^2} \, \hat{\boldsymbol{z}}$$

Here, the z-direction is the direction of current flow. At radii greater than R, the field is zero. We can calculate the magnetic field using the relation

$$\nabla \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Since the electric field points in the z-direction, we can deduce that the magnetic field will be circumferential, and use a circle as our "Amperian loop". We can then say:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 \varepsilon_0 \int \frac{\partial \boldsymbol{E}}{\partial t} \cdot d\boldsymbol{a}$$

Or,

$$B(2\pi s) = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} (\pi s^2)$$

Or,

$$B(2\pi s) = \mu_0 \varepsilon_0 \left[ \frac{I}{\varepsilon_0 \pi R^2} \right] \pi s^2$$

Or,

$$\boldsymbol{B} = \frac{\mu_0 I s}{2\pi R^2} \widehat{\boldsymbol{\phi}}$$

(b) The Poynting vector on the rim is given by

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} = \frac{1}{\mu_0} \left[ \frac{Q}{\varepsilon_0 \pi R^2} \hat{\boldsymbol{z}} \right] \times \left[ \frac{\mu_0 I}{2\pi R} \hat{\boldsymbol{\phi}} \right] = -\frac{QI}{2\pi^2 \varepsilon_0 R^3} \hat{\boldsymbol{s}}$$

(c) The total energy between the plates is given by

$$U = \frac{\varepsilon_0}{2} \int E^2 d\tau + \frac{1}{2\mu_0} \int B^2 d\tau$$
  
=  $\frac{\varepsilon_0}{2} \left[ \frac{Q}{\varepsilon_0 \pi R^2} \right]^2 [\pi R^2 d] + \frac{1}{2\mu_0} \left[ \frac{\mu_0 I}{2\pi R^2} \right]^2 (2\pi d) \int_0^R s^3 ds$   
=  $\frac{Q^2 d}{2\pi \varepsilon_0 R^2} + \frac{\mu_0 I^2 d}{16\pi}$ 

The rate of change of the energy is

$$\frac{dU}{dt} = \frac{QId}{\pi\varepsilon_0 R^2}$$

The energy flow is found from the Poynting vector

$$\int \boldsymbol{S} \cdot d\boldsymbol{a} = \frac{QI}{2\pi^2 \varepsilon_0 R^3} (2\pi Rd) = \frac{QId}{\pi \varepsilon_0 R^2}$$

## 8. Show that the standing wave

$$f(z,t) = Asin(kz)cos(kvt)$$

Satisfies the wave equation and express it as a sum of a wave traveling to the left and a wave traveling to the right.

Key:

$$\begin{array}{lll} \frac{\partial f}{\partial z} &=& Ak\cos(kz)\cos(kvt); \ \frac{\partial^2 f}{\partial z^2} = -Ak^2\sin(kz)\cos(kvt); \\ \frac{\partial f}{\partial t} &=& -Akv\sin(kz)\sin(kvt); \ \frac{\partial^2 f}{\partial t^2} = -Ak^2v^2\sin(kz)\cos(kvt) = v^2\frac{\partial^2 f}{\partial z^2}. \end{array}$$

Use the trig identity  $\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$  to write

$$f = \boxed{\frac{A}{2} \left\{ \sin[k(z+vt)] + \sin[k(z-vt)] \right\}},$$

which is of the form 9.6, with  $g = (A/2) \sin[k(z - vt)]$  and  $h = (A/2) \sin[k(z + vt)]$ .

Note that instead of the trig identity I can rewrite the sine and cosine function in the wave-function by complex exponential functions and then conclude that the given wave-function can be written as f(z-vt)+h(z+vt).

9. Using the exact amplitude coefficients of reflection and transmission (i.e. not assuming that u<sub>1</sub>=u<sub>2</sub>=u<sub>0</sub>) determine expressions for the reflection and transmission coefficients (ratio of intensity!) at <u>perpendicular incidence</u> and proof that R+T=1. Use the definition of the E-field and B-field axis for the incident, reflected, and transmitted waves depicted in the figure below. Assume that the z=0 plane forms the interface between vacuum (z<0) and the material (z>0).



Some of you proofed it for arbitrary angle of incidence.